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CHARACTERIZING CONGRUENCE PRESERVING FUNCTIONS $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ VIA RATIONAL POLYNOMIALS

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Abstract

Using a simple basis of rational polynomial-like functions $P_0, \ldots, P_{n-1}$ for the free module of functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, we characterize the subfamily of congruence preserving functions as the set of linear combinations of the products $\operatorname{lcm}(k)\, P_k$ where $\operatorname{lcm}(k)$ is the least common multiple of $2, \ldots, k$ (viewed in $\mathbb{Z}/m\mathbb{Z}$). As a consequence, when $n \geq m$, the number of such functions is independent of n.

1. Introduction

The notion of a congruence preserving function on rings of residue classes was introduced in Chen [3] and studied in Bhargava [1].

Definition 1.1. Let $m, n \geq 1$. A function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is said to be *congruence preserving* if for all d dividing m

$$\text{for all } a, b \in \{0, \ldots, n-1\} \quad a \equiv b \pmod{d} \text{ implies } f(a) \equiv f(b) \pmod{d}. \quad (1)$$

Remark 1.2. 1. If $n \in \{1, 2\}$ or $m = 1$ then every function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is trivially congruence preserving.

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2. Observe that since d is assumed to divide m, equivalence modulo d is a congruence on $(\mathbb{Z}/m\mathbb{Z}, +, \times)$. However, since d is not supposed to divide n, equivalence modulo d may not be a congruence on $(\mathbb{Z}/n\mathbb{Z}, +, \times)$.

Example 1.3. 1. For functions $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$, condition (1) reduces to the conditions $f(3) \equiv f(0) \pmod 3$, $f(4) \equiv f(1) \pmod 3$, $f(5) \equiv f(2) \pmod 3$.
2. For functions $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$, condition (1) reduces to $f(2) \equiv f(0) \pmod 2$, $f(3) \equiv f(1) \pmod 2$, $f(4) \equiv f(0) \pmod 4$, $f(5) \equiv f(1) \pmod 4$.

In this paper, we characterize congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

We denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N}$ the set of nonnegative integers (including zero).

Definition 1.4. The unary *lcm* function $\mathbb{N} \to \mathbb{N}$ maps 0 to 1 and $k \geq 1$ to the least common multiple of $1, 2, \ldots, k$.

A natural way to associate with each map from $\mathbb{N}$ to $\mathbb{Z}$ a map from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ is to restrict F to $\{0, \cdots, n-1\}$ and take its values modulo m.

Definition 1.5. With each map $F : \mathbb{N} \to \mathbb{Z}$, we associate the map $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ defined by $f = \pi_m \circ F \circ \iota_n$, where $\pi_m(x) = x \pmod m$, and $\iota_n(z)$ is the unique element of $\pi_n^{-1}(z) \cap \{0, \ldots, n-1\}$.

Definition 1.5 is best pictured by the commutativity of diagram (2).

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "F"] & \mathbb{Z} \arrow[d, "\pi_m"] \\
\mathbb{Z}/n\mathbb{Z} \arrow[u, "\iota_n"] \arrow[r, "f"] & \mathbb{Z}/m\mathbb{Z}
\end{tikzcd} \tag{2}$$

Applying Definition 1.5 to binomial coefficients, we obtain a basis of the $(\mathbb{Z}/m\mathbb{Z})$-module of functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

Proposition 1.6. *Let $P_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be associated with the $\mathbb{N} \to \mathbb{N}$ binomial function $x \mapsto \binom{x}{k}$. For every function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ there is a unique sequence $(a_0, \ldots, a_{n-1})$ of elements of $\mathbb{Z}/m\mathbb{Z}$ such that*

$$f \quad = \quad \sum_{k=0}^{k=n-1} a_k \, P_k \ . \tag{3}$$

In other words, the family $\{P_0, \ldots, P_{n-1}\}$ is a basis of the $(\mathbb{Z}/m\mathbb{Z})$-module of functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

Our main result can be stated as

Theorem 1.7. *A function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if, for each $k = 0, \ldots, n-1$, in equation* (3) *the coefficient a_k is a multiple of the residue of* $\operatorname{lcm}(k)$ *in $\mathbb{Z}/m\mathbb{Z}$.*

The paper is organized as follows.

Proposition 1.6 is proved in Section 2 where, after recalling Chen's notion of a polynomial function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ (cf. [3]), we extend it to a notion of a rational polynomial function.

The proof of our main result, Theorem 1.7, is given in Section 3. We adapt the techniques of our paper [2], exploiting similarities between Definition 1.1 and the condition studied in [2] for functions $f : \mathbb{N} \to \mathbb{Z}$ (namely, $x-y$ divides $f(x)-f(y)$ for all $x, y \in \mathbb{N}$). As a consequence of Theorem 1.7, the number of congruence preserving functions is independent of n for $n \geq m$ and even for $n \geq gpp(m)$ (the greatest prime power dividing m). Also, every congruence preserving function $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is a rational polynomial for a polynomial of degree strictly less than the minimum of n and $gpp(m)$.

In Section 4 we use our main theorem to count the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. We thus get an expression equivalent to that obtained by Bhargava in [1] and which makes apparent the fact that, for $n \geq gpp(m)$ (hence for $n \geq m$), this number depends only on m and is independent of n.

2. Representing Functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by Rational Polynomials

In [3, 1], congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ are introduced and studied together with an original notion of polynomial function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

Definition 2.1 (Chen [3])**.** A function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is *polynomial* if it is associated (in the sense of Definition 1.5) with a function $F : \mathbb{N} \to \mathbb{Z}$ given by a polynomial in $\mathbb{Z}[X]$.

Polynomial functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ are obviously congruence preserving. Are all congruence preserving functions polynomial? Chen [3] observed that this is not the case for some values of n, m, for instance $n = 6$, $m = 8$. He also proves that a stronger identity holds for infinitely many ordered pairs $\langle n, m\rangle$: *every function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is polynomial if and only n is not greater than the first prime factor of m* (in particular, this is the case when $n = m$ and m is prime, cf. Kempner [4]). Using counting arguments, Bhargava [1] characterizes the ordered pairs $\langle n, m\rangle$ such that every congruence preserving function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is polynomial.

Some polynomials in $\mathbb{Q}[X]$ (i.e., polynomials with rational coefficients) happen to map integers into integers.

Definition 2.2. For $k \in \mathbb{N}$, let $P_k \in \mathbb{Q}[X]$ be the following polynomial:

$$P_k(x) = \binom{x}{k} = \frac{\prod_{i=0}^{k-1}(x-i)}{k!}.$$

We will use the following examples later on:
$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x(x-1)/2$, $P_3(x) = x(x-1)(x-2)/6$, $P_4(x) = x(x-1)(x-2)(x-3)/24$, $P_5(x) = x(x-1)(x-2)(x-3)(x-4)/120$.

In [5], Pólya used the P_k's to give the following very elegant and elementary characterization of polynomials in $\mathbb{Q}[X]$ mapping integers to integers.

Theorem 2.3 (Pólya)**.** *A polynomial in* $\mathbb{Q}[X]$ *is integer-valued on* $\mathbb{Z}$ *if and only if it can be written as a* $\mathbb{Z}$*-linear combination of the polynomials* P_k*,* $k = 0, 1, 2, \ldots$.

It turns out that the representation of functions $\mathbb{N} \to \mathbb{Z}$ as $\mathbb{Z}$-linear combinations of the P_k's used in [2] also fits in the case of functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$: every such function is a $(\mathbb{Z}/m\mathbb{Z})$-linear combination of the P_k's.

Definition 2.4. 1. A function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is *rat-polynomial* if is associated in the sense of Definition 1.5 with some polynomial in $\mathbb{Q}[X]$.
2. The *degree* of a rat-polynomial function is the smallest degree of an associated polynomial in $\mathbb{Q}[X]$.
3. We denote by $P_k^{n,m}$ the rat-polynomial function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ associated with the polynomial P_k of Definition 2.2 in the sense of Definition 1.5. When there is no ambiguity, $P_k^{n,m}$ will be denoted simply as P_k.

Remark 2.5. In Definition 2.4, the polynomial *crucially depends* on the choice of representatives of elements of $\mathbb{Z}/n\mathbb{Z}$: e.g., for $n = m = 6$, $0 \equiv 6 \pmod 6$ but $0 = P_2(0) \not\equiv P_2(6) = 3 \pmod 6$. The chosen representatives for elements of $\mathbb{Z}/n\mathbb{Z}$ will always be $0, 1, \ldots, n-1$.

We now prove the representation result by the P_k's.

Proof of Proposition 1.6. Let us start with uniqueness. We have $f(0) = a_0$, and hence a_0 is $f(0)$. We have $f(1) = a_0 + a_1$, and hence $a_1 = f(1) - f(0)$. By induction, letting $Q_k = \sum_{\ell=0}^{\ell=k-1} a_\ell P_\ell$, and noting that $P_k(k) = 1$, we have $f(k) = Q_k(k) + a_k P_k(k) = Q_k(k) + a_k$, and hence $a_k = f(k) - Q_k(k)$. We thus are able to determine a_k in $\mathbb{Z}/m\mathbb{Z}$.

For existence, argue backwards to see that this sequence suits. ☐

Remark 2.6. The evaluation of $a_k\, P_k(x)$ in $\mathbb{Z}/m\mathbb{Z}$ has to be done as follows: for x an element of $\mathbb{Z}/n\mathbb{Z}$, we consider it as an element of $\{0, \ldots, n-1\} \subseteq \mathbb{N}$ and we evaluate $P_k(x) = \dfrac{1}{k!} \prod_{i=0}^{k-1}(x-i)$ as an element of $\mathbb{Z}$, then we consider the remainder modulo m, and finally we multiply the result by a_k in $\mathbb{Z}/m\mathbb{Z}$. For instance, for

$n = m = 8$, we have $4\,P_2(3) = 4 \times \dfrac{3\times 2}{2} = 4 \times 3 = 4$, but we might be tempted to evaluate it as $4\,P_2(3) = \dfrac{4\times 3\times 2}{2} = \dfrac{0}{2} = 0$, which does *not* correspond to our definition. However, dividing a_k by a factor of the denominator is allowed.

Corollary 2.7. *1. Every function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial with degree less than n.*
2. The family of rat-polynomial functions $\{P_k \mid k = 0, 1, \ldots, n-1\}$ is a basis of the $(\mathbb{Z}/m\mathbb{Z})$-module of functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

Example 2.8. The function $f\colon \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ such that $f(0) = 0$, $f(1) = 3$, $f(2) = 4$, $f(3) = 3$, $f(4) = 0$, $f(5) = 1$, is represented by the rational polynomial $P_f(x) = 3x + 4\,\dfrac{x(x-1)}{2}$ which can be simplified to $P_f(x) = 3x - x(x-1)$ on $\mathbb{Z}/6\mathbb{Z}$.

Example 2.9. The function $f\colon \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ given by Chen [3] as a non-polynomial congruence preserving function, namely the function such that $f(0) = 0$, $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 4$, $f(5) = 7$, is represented by the rational polynomial with coefficients $a_0 = 0$, $a_1 = 3$, $a_2 = 6$, $a_3 = 2$, $a_4 = 4$, $a_5 = 4$. Thus,

$$\begin{aligned} f(x) &= 3x + 6\,\frac{x(x-1)}{2} + 2\,\frac{x(x-1)(x-2)}{2} + 4\,\frac{x(x-1)(x-2)(x-3)}{8} \\ &\quad + 4\,\frac{x(x-1)(x-2)(x-3)(x-4)}{8} \\ &= 3x + 3x(x-1) + x(x-1)(x-2) + \frac{x(x-1)(x-2)(x-3)}{2} \\ &\quad + \frac{x(x-1)(x-2)(x-3)(x-4)}{2}. \end{aligned}$$

3. Characterizing Congruence Preserving Functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$

Congruence preserving functions $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ can be characterized by a simple condition on the coefficients of the rat-polynomial representation of f given in Proposition 1.6.

3.1. Proof of Theorem 1.7

For proving Theorem 1.7 we will need some relations involving binomial coefficients and the unary lcm function; these relations are stated in the next three lemmata. The proofs are elementary but technical and can be found in our paper [2].

Lemma 3.1. *If $0 \le n - k < p \le n$ then p divides $\mathrm{lcm}(k)\binom{n}{k}$ in $\mathbb{N}$.*

Lemma 3.2. *If $k \le b$ then n divides $A^n_{k,b} = \mathrm{lcm}(k)\left(\binom{b+n}{k} - \binom{b}{k}\right)$ in $\mathbb{N}$.*

The following is an immediate consequence of Lemma 3.2 (set $a = b + n$).

Lemma 3.3. *If* $a \geq b$ *and* $k \leq b$, *then* $a - b$ *divides* $\operatorname{lcm}(k)\left(\binom{a}{k} - \binom{b}{k}\right)$ *in* $\mathbb{N}$.

Besides these lemmata which deal with divisibility on integers, we shall use a classical result in $\mathbb{Z}/m\mathbb{Z}$. For $x, y \in \mathbb{Z}$ we say x *divides* y *in* $\mathbb{Z}/m\mathbb{Z}$ if and only if the residue class of x divides the residue class of y in $\mathbb{Z}/m\mathbb{Z}$.

Lemma 3.4. *Let* $1 \leq c_1, \ldots, c_k \leq m$ *and let* c *be their least common multiple in* $\mathbb{N}$. *If* $c_1, \ldots, c_k$ *all divide* a *in* $\mathbb{Z}/m\mathbb{Z}$ *then so does* c.

Proof. It suffices to consider the case $k = 2$ since the passage to any k is done via a straightforward induction. Let $c = c_1 b_1 = c_2 b_2$ with b_1, b_2 coprime. Let t, u be such that $a = c_1 t = c_2 u$ in $\mathbb{Z}/m\mathbb{Z}$. Then $a \equiv c_1 t \equiv c_2 u \pmod{m}$. Using Bézout's identity, let $\alpha, \beta \in \mathbb{Z}$ be such that $\alpha b_1 + \beta b_2 = 1$. Then $c(t\alpha + u\beta) = c_1 b_1 t\alpha + c_2 b_2 u\beta \equiv a\alpha b_1 + a\beta b_2 \pmod{m}$, and hence $c(t\alpha + u\beta) \equiv a \pmod{m}$, proving that c divides a in $\mathbb{Z}/m\mathbb{Z}$. ☐

Proof of the "only if" part of Theorem 1.7. Assume $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is congruence preserving and consider its decomposition $f(x) = \sum_{k=0}^{n-1} a_k P_k(x)$ given by Proposition 1.6. We show that $\operatorname{lcm}(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$ for all $k < n$. The cases $k = 0$ and $k = 1$ are trivial since $\operatorname{lcm}(0) = \operatorname{lcm}(1) = 1$.

Claim 1. *For all* $2 \leq k < n$, k *divides* a_k *in* $\mathbb{Z}/m\mathbb{Z}$.

Proof. Recall that $f(k) = \sum_{i=0}^{n-1} a_i \binom{k}{i} = \sum_{i=0}^{k} a_i \binom{k}{i}$ since $\binom{k}{i} = 0$ for $i > k$. We argue by induction on $k \geq 2$.
`Base case` $k = 2$. If 2 does not divide m then 2 and m are coprime, and hence 2 is invertible and divides a_2 in $\mathbb{Z}/m\mathbb{Z}$. Assume 2 divides m. As 2 divides $2 - 0$ and f is congruence preserving, 2 also divides $f(2) - f(0) = 2a_1 + a_2$, and hence 2 divides a_2.
`Inductive step.` Let $2 < k < n - 1$. The inductive hypothesis ensures that ℓ divides a_ℓ in $\mathbb{Z}/m\mathbb{Z}$ for every $\ell \leq k$. Let $a_\ell \equiv \ell q_\ell \pmod{m}$ for $0 \leq \ell \leq k$. We prove that $k + 1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$. First, observe that

$$\begin{aligned}
f(k+1) - f(0) &= (k+1)a_1 + \left(\sum_{i=2}^{k} \binom{k+1}{i} a_i\right) + a_{k+1} \\
&\equiv (k+1)a_1 + \left(\sum_{i=2}^{k} \binom{k+1}{i} i q_i\right) + a_{k+1} \pmod{m} \\
f(k+1) - f(0) &= (k+1)a_1 + \left(\sum_{i=2}^{k} (k+1) \binom{k}{i-1} q_i\right) + \alpha m + a_{k+1} \qquad (4)
\end{aligned}$$

for some α. Let $d = gcd(k+1, m)$. Since d divides m and $k+1-0$ and f is congruence preserving, d also divides $f(k+1) - f(0)$. Using equality (4), we see that d divides the last term a_{k+1} of the sum. Using Bézout's identity, let u, v be such that $u(k+1) + vm = d$. Then $u(k+1) \equiv d \pmod{m}$, and hence $k+1$ divides d in $\mathbb{Z}/m\mathbb{Z}$. Since d divides a_{k+1}, we conclude that $k+1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$. ☐

Claim 2. *(i) For all* $2 \le p \le k < n$, p *divides* a_k *in* $\mathbb{Z}/m\mathbb{Z}$.
(ii) For all $2 \le k < n$, $\operatorname{lcm}(k)$ *divides* a_k *in* $\mathbb{Z}/m\mathbb{Z}$.

Proof. Assertion *(ii)* is a direct application of Lemma 3.4 and assertion *(i)*. We prove *(i)* by induction on $p \ge 2$. Both the base case and the inductive step of this induction are proved by induction on k.

`Base case` $p = 2$. We have to prove that 2 divides a_k for all $k \ge 2$. If 2 does not divide m, then 2 is invertible and divides all numbers in $\mathbb{Z}/m\mathbb{Z}$. Assume now that 2 divides m. We argue by induction on $k \ge 2$.

`Base case.` Apply Claim 1: 2 divides a_2.

`Inductive step.` Let $k < n-1$. Assuming that 2 divides a_i for all $2 \le i \le k$, we prove that 2 divides a_{k+1}. Two cases can occur.

`Subcase 1:` $k+1$ `is odd.` Then 2 divides k and hence, by congruence preservation, 2 divides $f(k+1) - f(1)$. As $f(k+1) - f(1) = ka_1 + \left(\sum_{i=2}^{k} a_i \binom{k+1}{i}\right) + a_{k+1}$, and 2 divides k and also, by the induction hypothesis, 2 divides a_i for $2 \le i \le k$, we see that 2 divides a_{k+1}.

`Subcase 2:` $k+1$ `is even.` By congruence preservation, 2 divides $f(k+1) - f(0) = (k+1)a_1 + \left(\sum_{i=2}^{k} a_i \binom{k+1}{i}\right) + a_{k+1}$. Since 2 divides $k+1$ and a_i for $2 \le i \le k$ (induction hypothesis), we infer that 2 divides a_{k+1}.

`Inductive step`. Let $2 \le p < n-1$ and assume that

$$\text{for all } q \le p \text{ and all } \ell \text{ such that } q \le \ell < n,\ q \text{ divides } a_\ell \text{ in } \mathbb{Z}/m\mathbb{Z}. \tag{5}$$

By induction on $k \ge p+1$, we prove that $p+1$ divides a_k for all k such that $p+1 \le k < n$.

`Base case` $k = p+1$. Apply Claim 1: $p+1$ divides a_{p+1}.

`Inductive step.` Let $k < n-1$. Assuming that $p+1$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$ for all i

such that $p+1 \le i \le k$, we prove that $p+1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$. We have

$$f(k+1) - f(k-p) = \sum_{i=1}^{k-p} a_i \left(\binom{k+1}{i} - \binom{k-p}{i} \right) + \left(\sum_{i=k+1-p}^{k} a_i \binom{k+1}{i} \right) + a_{k+1} \qquad (6)$$

We first look at the terms of the first sum on the right side of (6) corresponding to $1 \le i \le p$. Applying (5) with $\ell = i$, we see that q divides a_i in $\mathbb{Z}/m\mathbb{Z}$ for all $q \le \min(p, i) = i$. Using Lemma 3.4, we conclude that $\operatorname{lcm}(i)$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$. Observing that $(k+1) = (k-p)+(p+1)$, we can apply Lemma 3.2 (with $k-p$, $p+1$ and i in place of b, n and k) and conclude that $p+1$ divides $\operatorname{lcm}(i) \left(\binom{k+1}{i} - \binom{k-p}{i} \right)$ in $\mathbb{N}$. Thus, $p+1$ divides $a_i \left(\binom{k+1}{i} - \binom{k-p}{i} \right)$ in $\mathbb{Z}/m\mathbb{Z}$.

We now turn to the terms of the first sum on the right side of (6) corresponding to $p+1 \le i \le k-p$ (if there are any). Each of these terms is divisible by $p+1$ in $\mathbb{Z}/m\mathbb{Z}$, because the induction hypothesis on k ensures that $p+1$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$ whenever $p+1 \le i \le k$.

Consider next the terms of the second sum on the right side of (6). For those terms corresponding to values of i such that $p+1 \le i \le k$, divisibility by $p+1$ in $\mathbb{Z}/m\mathbb{Z}$ follows from the fact that, by the induction hypothesis on k, $p+1$ divides a_i. It remains to look at the terms associated with the i's such that $k+1-p \le i \le p$ (there are such i's in case $k+1-p < p+1$). For such i's we have $0 \le (k+1)-i \le (k+1)-p < p+1 \le k+1$ and Lemma 3.1 (used with $k+1, i$ and $p+1$ in place of n, k and p) implies that $p+1$ divides $\operatorname{lcm}(i)\binom{k+1}{i}$. Now, for such i's, the induction hypothesis (5) on p shows that $\operatorname{lcm}(i)$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$. A fortiori, $p+1$ divides $a_i\binom{k+1}{i}$ in $\mathbb{Z}/m\mathbb{Z}$.

Let $d = gcd(p+1, m)$. As $p+1$ divides in $\mathbb{Z}/m\mathbb{Z}$ all terms of the two sums on the right side of (6) so does d. Since d divides m and $k+1-(k-p) = p+1$ and f is congruence preserving, d also divides $f(k+1) - f(k-p)$. Using equality (6), we conclude that d divides in $\mathbb{Z}/m\mathbb{Z}$ the last term a_{k+1}. Using Bézout's identity, let u, v be such that $u(p+1) + vm = d$. Then $u(p+1) \equiv d \pmod{m}$, and hence $p+1$ divides d in $\mathbb{Z}/m\mathbb{Z}$. As d divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$, we conclude that $p+1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$.

This ends the proof of the induction in the inductive step, and hence also the proof of Claim 2 and of the "only if" part of the Theorem. □

Proof of the "if" part of Theorem 1.7. Assume $f = \sum_{k=0}^{k=n-1} a_k P_k$ and that all of the a_k's are divisible by $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$. We can write f in the form $f(n) = \sum_{k=0}^{n} c_k \operatorname{lcm}(k)\binom{n}{k}$. We prove that f is congruence preserving, i.e., if $0 \le b < a \le$

$n-1$ and d divides both m and $a-b$ then d also divides $f(a)-f(b)$. Observe that

$$f(a)-f(b)=\left(\sum_{k=0}^{b}c_k\mathrm{lcm}(k)\left(\binom{a}{k}-\binom{b}{k}\right)\right)+\sum_{k=b+1}^{a}c_k\mathrm{lcm}(k)\binom{a}{k}.$$

By Lemma 3.3, $a-b$ divides each term of the first sum. Consider the terms of the second sum. For $b+1\le k\le a$, we have $0\le a-k<a-b\le a$ and Lemma 3.1 (used with a,k and $a-b$ in place of n,k and p) shows that $a-b$ divides $\mathrm{lcm}(k)\dbinom{a}{k}$. Thus, $a-b$ divides $f(a)-f(b)$. ☐

3.2. On a Family of Generators

We now sharpen the degree of the rat-polynomial representing a congruence preserving function $\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$. We first state some properties of the lcm function in $\mathbb{N}$.

Lemma 3.5. *Let $m\ge 1$ be an integer with prime factorization $m=p_1^{\alpha_1}\cdots p_\ell^{\alpha_\ell}$. Then $\mathrm{lcm}(k)=u\prod_{i=1}^{\ell}p_i^{\alpha_{i,k}}$, where u is coprime with m and $\alpha_{i,k}=\max\{\beta_i\mid p_i^{\beta_i}\le k\}$.*

Definition 3.6. Let $m\ge 1$ be an integer with prime factorization $m=p_1^{\alpha_1}\cdots p_\ell^{\alpha_\ell}$. We let $gpp(m)=\max\left\{p_i^{\alpha_i}\mid i\in\{1,\ldots,\ell\}\right\}$ be the greatest power of prime dividing m in $\mathbb{N}$.

Lemma 3.7. *The number $gpp(m)$ is the least integer k such that m divides* $\mathrm{lcm}(k)$.

Example 3.8. We have $gpp(8)=8$, $gpp(12)=4$ and $gpp(14)=7$. The successive values of the residues in $\mathbb{Z}/m\mathbb{Z}$ of $\mathrm{lcm}(k)$ are

k	1	2	3	4	5	6	7	8
$\mathrm{lcm}(k)$ in $\mathbb{Z}/8\mathbb{Z}$	1	2	2	4	4	4	4	0
$\mathrm{lcm}(k)$ in $\mathbb{Z}/12\mathbb{Z}$	1	2	6	0	0	0	0	0
$\mathrm{lcm}(k)$ in $\mathbb{Z}/14\mathbb{Z}$	1	2	6	12	4	4	0	0

.

For all $\ell\ge gpp(m)$, $\mathrm{lcm}(\ell)$ is zero in $\mathbb{Z}/m\mathbb{Z}$.

Remark 3.9. 1. Either $gpp(m)=m$ or $gpp(m)\le m/2$.
2. In general, $gpp(m)$ is greater than $\lambda(m)$, the least k such that m divides $k!$ (a function considered in [3]): for $m=8$, $gpp(m)=8$ whereas $\lambda(m)=4$.

Using Lemma 3.7, we can get a better version of Theorem 1.7.

Theorem 3.10. *A function $f\colon\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if it is associated in the sense of Definition 1.5 with a rational polynomial $P=\sum_{k=0}^{d-1}a_k\binom{x}{k}$ where $d=\min(n,gpp(m))$ and such that lcm(k) divides a_k in $\mathbb{Z}/m\mathbb{Z}$ for all $k<d$.*

Proof. For $k \geq gpp(m)$, m divides $lcm(k)$ hence the coefficient a_k is 0. $\square$

Theorem 3.11. *(i) Every congruence preserving function* $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ *is rat-polynomial with degree less than* $gpp(m)$.
(ii) The family of rat-polynomial functions

$$\mathcal{F} = \{lcm(k)P_k \mid 0 \leq k < \min(n, gpp(m))\}$$

generates the set of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.
(iii) $\mathcal{F}$ *is a basis of the set of congruence preserving functions if and only if* m *has no prime divisor* $p < \min(n, m)$ *(in case* $n \geq m$ *this means that* m *is prime).*

Proof. Assertions *(i)* and *(ii)* are restatements of Theorem 3.10. Let us prove *(iii)*.

"Only If" part. Asssume m has a prime divisor $p < \min(n, m)$ and let p be the least one. Then $\mathrm{lcm}(p) = pa$ with a coprime with m, and hence $\mathrm{lcm}(p) \neq 0$ in $\mathbb{Z}/m\mathbb{Z}$. Since $P_p(p) = 1$ this shows that $\mathrm{lcm}(p)\, P_p$ is not the null function. However $(m/p)\, \mathrm{lcm}(p) = 0$ in $\mathbb{Z}/m\mathbb{Z}$, and hence $(m/p)\, \mathrm{lcm}(p)\, P_p$ is the null function. As $(m/p) \neq 0$ in $\mathbb{Z}/m\mathbb{Z}$, this proves that $\mathcal{F}$ cannot be a basis.

"If" part. Assume that m has no prime divisor $p < \min(n, m)$. We prove that $\mathcal{F}$ is $(\mathbb{Z}/m\mathbb{Z})$-linearly independent. Suppose that the $(\mathbb{Z}/m\mathbb{Z})$-linear combination $L = \sum_{k=0}^{\min(n,gpp(m))-1} a_k\, \mathrm{lcm}(k)\, P_k$ is the null function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. By induction on $k = 0, \ldots, \min(n, gpp(m)) - 1$ we prove that $a_k = 0$.

- *Basic cases* $k = 0, 1$. From $L(0) = a_0$ and $L(1) = a_0 + a_1$ we deduce $a_0 = a_1 = 0$.
- *Induction step.* Assuming $k \geq 2$ and $a_i = 0$ for $i = 0, \ldots, k - 1$, we prove that $a_k = 0$. Observe that $P_\ell(k) = \binom{k}{\ell} = 0$ for $k < \ell < n$. Since $a_i = 0$ for $i = 0, \ldots, k-1$, and $P_k(k) = 1$ we get $L(k) = a_k\, \mathrm{lcm}(k)$. As $k < \min(n, gpp(m)) \leq \min(n, m)$ and m has no prime divisor $p < \min(n, m)$, the numbers $\mathrm{lcm}(k)$ and m are coprime. Thus, $\mathrm{lcm}(k)$ is invertible in $\mathbb{Z}/m\mathbb{Z}$ and equality $L(k) = a_k\, \mathrm{lcm}(k) = 0$ implies $a_k = 0$. $\square$

4. Counting Congruence Preserving Functions

We now compute the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. As two different rational polynomials correspond to different functions by Proposition 1.6 (uniqueness of the representation by a rational polynomial), the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is equal to the number of polynomials representing them.

Proposition 4.1. *Let* $CP(n, m)$ *be the number of congruence preserving functions* $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. *Let* $m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ *be the decomposition of* m *in powers of* *primes. Let* $\mathcal{I} = \{i \mid p_i^{e_i} < gpp(m)\}$ *and* $\mathcal{J} = \{i \mid p_i^{e_i} \geq gpp(m)\}$*. Then*

$$CP(n,m) = \begin{cases} p_1^{p_1+p_1^2+\cdots+p_1^{e_1}} \times \cdots \times p_\ell^{p_\ell+p_\ell^2+\cdots+p_\ell^{e_\ell}} & \text{if } n \geq gpp(m), \\ \prod_{i\in\mathcal{I}} p_i^{p_i+p_i^2+\cdots+p_i^{e_i}} \times \prod_{i\in\mathcal{J}} p_i^{p_i+p_i^2+\cdots+p_i^{\lfloor \log_p n\rfloor}+n(e-\lfloor \log_p n\rfloor)} & \text{if } n < gpp(m). \end{cases}$$

Equivalently, writing $E(p,\alpha)$ *instead of* p^α *for better readability, we have*

$$CP(n,m) = \begin{cases} \prod_{i=1}^{\ell} E(p_i, \sum_{k=1}^{e_i} p_i^k) & \text{if } n \geq gpp(m), \\ \prod_{i\in\mathcal{I}} E(p_i, \sum_{k=1}^{e_i} p_i^k) \times \prod_{i\in\mathcal{J}} E(p_i, (\sum_{k=1}^{\lfloor \log_p n\rfloor} p_i^k) + n(e - \lfloor \log_p n\rfloor)) & \text{if } n < gpp(m). \end{cases}$$

Corollary 4.2. *For* $n \geq gpp(m)$*,* $CP(n,m)$ *does* not *depend on* n*.*

Proof of Proposition 4.1. By Theorem 3.10, we must count the number of n-tuples of coefficients $(a_0, \ldots, a_{n-1})$, with, for $k = 0, \ldots, n-1$, a_k being a multiple of $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$.

Claim 1. *For* $m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$*, for all* n*,* $CP(n,m) = \prod_{i=1}^{\ell} CP(n, p_i^{e_i})$*.*

Proof of Claim 1. Let $E(r,k)$ be the set of multiples in $\mathbb{Z}/r\mathbb{Z}$ of $\mathrm{lcm}(k)$ and $\lambda(r,k)$ be the cardinal of $E(r,k)$. The Chinese remainder theorem shows that the map $\rho : z \mapsto \big(z \pmod{p_i^{e_i}}\big)_{i=1,\ldots,\ell}$ is an isomorphism and also that ρ maps the set $E(m,k)$ onto the Cartesian product $P = \prod_{i=1}^{\ell} E(p_i^{e_i}, k)$. Indeed, let $(t_i)_{i=1,\ldots,\ell} \in P$. For each $i = 1, \ldots, \ell$, there is $0 \leq q_i < p_i^{e_i}$ such that $t_i \equiv q_i \,\mathrm{lcm}(k) \pmod{p_i^{e_i}}$. Applying the Chinese remainder theorem, there are $0 \leq t, q < m$ such that $t \equiv t_i \pmod{p_i^{e_i}}$ and $q \equiv q_i \pmod{p_i^{e_i}}$. Then $t \equiv q \,\mathrm{lcm}(k) \pmod{m}$, and hence $\rho(t) = (t_i)_{i=1,\ldots,\ell}$. This proves that $\lambda(m,k) = \prod_{i=1}^{\ell} \lambda(p_i^{e_i}, k)$ for each k. Thus, the number $CP(n,m)$ of n-tuples $(a_0, \ldots, a_{n-1})$ such that $lcm(k)$ divides a_k is equal to

$$CP(n,m) = \prod_{k<n} \lambda(m,k) = \prod_{k<n} \prod_{i=1}^{\ell} \lambda(p_i^{e_i}, k) = \prod_{i=1}^{\ell} \prod_{k<n} \lambda(p_i^{e_i}, k) = \prod_{i=1}^{\ell} CP(n, p_i^{e_i}). \quad \square$$

Claim 1 reduces the problem to that of counting the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p_i^{e_i}\mathbb{Z}$. We will use Theorem 3.10 to this end.

Claim 2. *Letting* $\ell = \lfloor \log_p n \rfloor$ *(and using the* $E(p,\alpha)$ *notation for* p^α*), we have*

$$CP(n, p^e) = \begin{cases} E(p, p + p^2 + \cdots + p^e) & \text{if } n \geq p^e, \\ E(p, p + p^2 + \cdots + p^\ell + (e-\ell)n) & \text{if } p^\ell \leq n < p^e. \end{cases}$$

Proof of Claim 2. By Theorem 3.10, as $gpp(p^e) = p^e$, letting $\nu = \inf(n, p^e)$, we have $CP(n, p^e) = CP(\nu, p^e) = \prod_{k=0}^{\nu-1} \lambda(p^e, k)$. As we noted in the proof of Claim 1, for

$p^j \leq k < p^{j+1}$, the order $\lambda(p^e, k)$ of the subgroup generated by $lcm(k)$ in $\mathbb{Z}/p^e\mathbb{Z}$ is p^{e-j}, and there are $p^{j+1} - p^j$ such k's. For $k = 0$, $\mathrm{lcm}(0) = 1$ yields $\lambda(p^e, 0) = p^e$.

- If $n \geq p^e$ then $CP(n, p^e) = CP(p^e, p^e) = p^e \prod_{j=0}^{e-1} \prod_{k=p^j}^{p^{j+1}-1} p^{e-j} = p^M$ with

$$M \quad = \quad e + \sum_{j=0}^{e-1} (e-j)(p^{j+1} - p^j) = p + p^2 + \cdots + p^e$$

- If $n < p^e$ then $p^\ell \leq n < p^e$ and

$$\begin{aligned} CP(n, p^e) &= \textstyle\prod_{k=0}^{n-1} \lambda(p^e, k) \\ &= p^e (\textstyle\prod_{j=0}^{\ell-1} \prod_{k=p^j}^{p^{j+1}-1} p^{e-j})(\prod_{k=p^\ell}^{n-1} p^{e-\ell}) = p^M \text{ with} \\ M &= e + \sum_{j=0}^{\ell-1} (e-j)(p^{j+1} - p^j) + \sum_{k=p^\ell}^{n-1} (e-\ell) \\ &= (e-\ell)p^\ell + (p + p^2 + \cdots + p^\ell) + (n - p^\ell)(e-\ell) \\ &= (p + p^2 + \cdots + p^\ell) + n(e-\ell) \qquad \square \end{aligned}$$

This finishes the proof of Proposition 4.1. $\square$

Remark 4.3. In [1] the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ is shown to be equal to $E(p, en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\})$. For $p^i \leq k < p^{i+1}$, we have $\lfloor \log_p k \rfloor = i$, and hence $\min\{e, \lfloor \log_p k \rfloor\} = \lfloor \log_p k \rfloor$ for $k \leq p^e$, and $\min\{e, \lfloor \log_p k \rfloor\} = e$ for $k \geq p^e$. Thus, we have

- if $n \geq p^e$, then $\sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\} = \sum_{k=1}^{p^e-1} \lfloor \log_p k \rfloor + \sum_{k=p^e}^{n-1} e = \sum_{j=0}^{e-1} j(p^{j+1} - p^j) + e(n - p^e)$ $= -(p + \cdots + p^e) + ep^e + e(n - p^e)$, and hence $en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\} = p + \cdots + p^e$. This coincides with our counting in Claim 2.
- if $n < p^e$, and $l = \lfloor \log_p n \rfloor$, then, similarly, $\sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = \sum_{k=1}^{\ell-1} \lfloor \log_p k \rfloor + \sum_{k=l}^{n-1} \lfloor \log_p k \rfloor = \sum_{j=0}^{\ell-1} j(p^{j+1} - p^j) + \ell(n - p^\ell) =$ $-(p + \cdots + p^\ell) + n\ell$, and hence $en - \sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = p + \cdots + p^\ell + (e-\ell)n$. Again, this coincides with our counting in Claim 2.

5. Conclusion

We proved that the rational polynomials $\mathrm{lcm}(k)\, P_k$ generate the $\mathbb{Z}/m\mathbb{Z}$ submodule of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. When n is larger than the greatest prime power dividing m, the number of functions in this submodule is independent of n. An open problem is the existence of a basis of this submodule.

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