ELSEVIER Annals of Pure and Applied Logic 89 (1997) 45–52

ANNALS OF PURE AND APPLIED LOGIC

Theories of generalized Pascal triangles [1]

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0. Introduction

In this talk I will summarize some of my results concerning first order theories of generalized Pascal triangles (GPT). Most of them are proved in [5–12]. My talk will consist of the following parts:

(1) *Definition of GPT and a motivation.*
Generalized Pascal triangles (GPT) will be special binary partial operations on the set $\mathbb{N}$ of nonnegative integers.

(2) *Structures associated to GPT.*
Several natural ways will be suggested how to associate a mathematical structure (a partial algebra or a relational system) to a GPT. By a theory of GPT we shall mean the first order theory of an associated structure.

(3) *An example* of a GPT for which decidability of its theory surprisingly strongly depends on the chosen structure.

(4) *Decidability results for theories of GPT modulo an integer.*
In this case decidability or undecidability of the theory of a GPT mainly depends on the factorization of the modulus.

(5) *Decidability results for theories of GPT of small algebras.*
The smallest cardinalities of the underlying algebras will be shown where the theories of corresponding GPT can be undecidable.

1. Definition of GPT and a motivation

The notion of Pascal's triangle was generalized in many various ways; these ways can be roughly divided into arithmetical and algebraical ones. We shall deal with the later ones; they were considered in [4]. Many arithmetical generalizations are considered in [2]. The classification is neither strict nor exclusive: Pascal's triangles modulo n belong to both types.

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[1] This work was partially supported by LLAIC1, Université d'Auvergne, Clermont 1, and by Grant 1227 of Slovak Academy of Sciences.

PII S0168-0072(97)00006-7

Generalized Pascal triangles (GPT) are mappings of the sets

$$\mathbb{D}_n = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x + y \geqslant n - 1\},$$

$n > 0$, into finite sets associated to some finite algebras analogously as the classical Pascal triangle can be associated to the (infinite) algebra $\langle \mathbb{N}; +, 0\rangle$ and "the word" 1 (of length 1).

To every finite algebra $\mathscr{A} = \langle \mathbf{A}; *, \mathrm{o}\rangle$ (with $\mathbf{A} \subseteq \mathbb{N}$ to avoid some technical problems) such that $\mathrm{o} * \mathrm{o} = \mathrm{o}$ and every word $w \in \mathbf{A}^+$ we shall construct $G = \mathrm{GPT}(\mathscr{A}, w)$ as follows. The initial word w is written down into the initial row (with spaces between its letters). We imagine that the constant o is written in all positions before and after w. Every further row is formed from the previous one by the operation $*$ analogously as $+$ is used in the classical Pascal triangle. At the margins o is used analogously as 0 in the classical Pascal triangle.

More formally, if $w = w_0 \ldots w_{|w|-1}$ then the function $G = \mathrm{GPT}(\mathbf{A}, w)$ will be defined by the following formula:

$$G(x, y) = \begin{cases} \text{undefined} & \text{if } x + y < |w| - 1, \\ w_x & \text{if } x + y = |w| - 1, \\ \mathrm{o} * G(0, y - 1) & \text{if } x = 0,\ y \geqslant |w|, \\ G(x - 1, 0) * \mathrm{o} & \text{if } y = 0,\ x \geqslant |w|, \\ G(x - 1, y) * G(x, y - 1) & \text{if } x + y \geqslant |w|,\ x > 0,\ y > 0. \end{cases}$$

The system of coordinates is chosen so that the whole GPT lies in the first (i.e., "positive") quadrant of the plane.

For example, if $\mathbf{Z}_n$ is the additive group modulo n then $B_n = \mathrm{GPT}(\mathbf{Z}_n, 1)$ is the Pascal triangle modulo n; we have

$$B_n(x, y) = \binom{x + y}{x} \operatorname{MOD} n \qquad \text{for all } x, y \in \mathbb{N}.$$

The cases $n = 2$ and $n = 3$ are displayed in Fig. 1.

And now several words to motivation. Physicists sometimes use cellular automata (CA) to simulate some physical processes. Of course, 3-dimensional CA will provide the most realistic models, but sometimes also one-dimensional CA suffice. GPT correspond to computations of one-dimensional CA from finite initial configurations. The computations of three-dimensional CA can be considered in $(3 + 1)$-dimensional discrete space-time. Analogously GPT can be considered in $(1 + 1)$-dimensional discrete space-time. In the formal definition of GPT the sum $x + y$ corresponds to the time coordinate and $x - y$ to the space coordinate. The domain of a GPT will be considered as a "light cone", which contains the whole interesting part of the CA computation; outside of it all automata states are o (or zero). In $(1 + 1)$-dimensional space-time the light cone is determined by two linear inequalities (and, maybe, the third one: time $\geqslant 0$). In the system of coordinates introduced here these inequalities are the simplest possible: $x \geqslant 0$ and $y \geqslant 0$. This is also advantageous when we associate mathematical

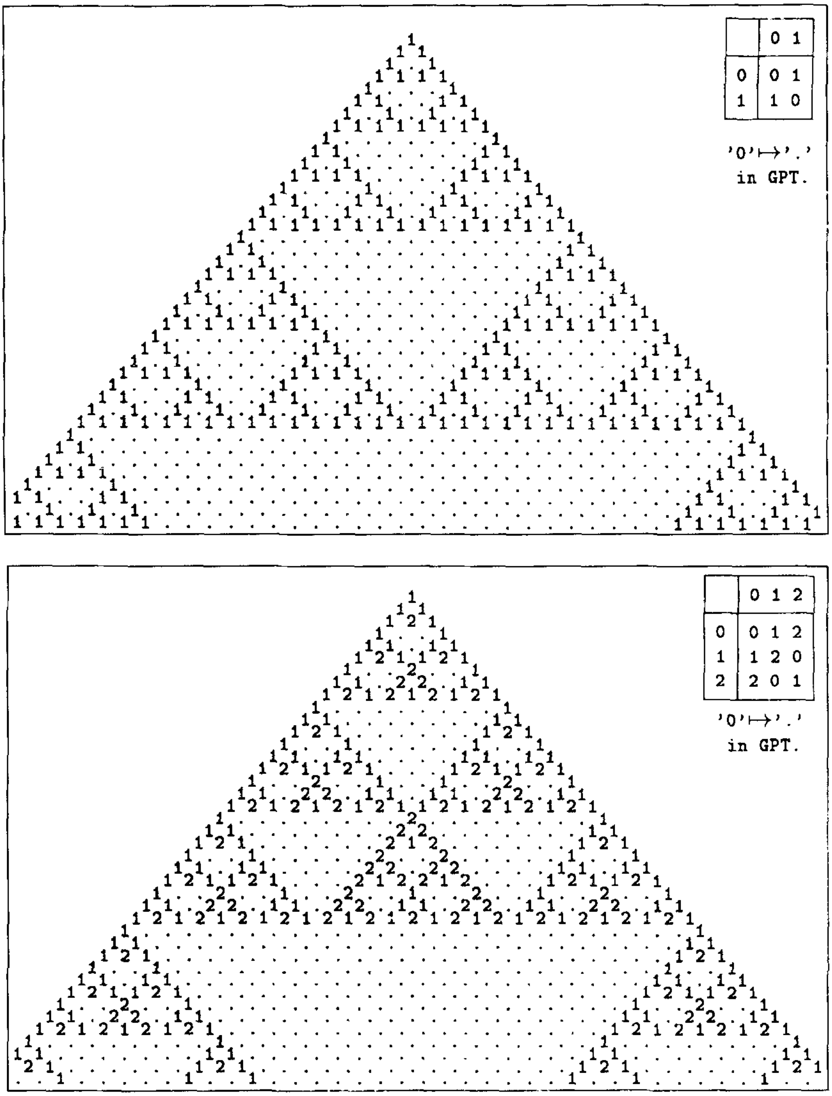

Fig. 1.

structures to the CA computations: we can use the set $\mathbb{N}$ instead the set of all integers (and the inequalities need not be explicitly considered).

2. Structures associated to GPT

By the definition, a GPT G is a partial binary operation on the set $\mathbb{N}$. There are many ways how to form a whole mathematical structure from it. We can consider the following possibilities:

(1) The simplest way is to consider the partial groupoid $\langle \mathbb{N}; G\rangle$. Technical problems with partiality can be easily solved (remember that the domain of G is a cofinite subset of $\mathbb{N} \times \mathbb{N}$).

(2) We can consider the relational structure $\langle \mathbb{N}; \mathrm{EqG}\rangle$, where EqG corresponds to the equivalence relation induced by G, i.e.

$$\mathrm{EqG} = \{(x,y,z,w) \in \mathrm{Dom}(G) \times \mathrm{Dom}(G) \mid G(x,y) = G(z,w)\}.$$

(3) We can consider the two-sorted relational structure $\langle \mathbb{N}^{\mathrm{x}}, \mathbb{N}^{\mathrm{y}}; \mathrm{EqG}\rangle$ where $\mathbb{N}^{\mathrm{x}} = \mathbb{N}^{\mathrm{y}} = \mathbb{N}$ and EqG is considered as a subset of $\mathbb{N}^{\mathrm{x}} \times \mathbb{N}^{\mathrm{y}} \times \mathbb{N}^{\mathrm{x}} \times \mathbb{N}^{\mathrm{y}}$. (The superscripts only distinguish the universes. Generally speaking the identical mapping of $\mathbb{N}^{\mathrm{x}}$ onto $\mathbb{N}^{\mathrm{y}}$ is not definable in the mentioned structure. If it is then distinguishing of types is unnecessary in essential.)

(4) Analogously the three-sorted partial algebra $\langle \mathbb{N}^{\mathrm{x}}, \mathbb{N}^{\mathrm{y}}, \mathbf{A}; G\rangle$ where G is considered as a map of (a subset of) $\mathbb{N}^{\mathrm{x}} \times \mathbb{N}^{\mathrm{y}}$ into $\mathbf{A}$. The third sort $\mathbf{A}$ can be deleted (and replaced by $\mathbb{N}^{\mathrm{x}}$ where necessary).

(5) In every case (1)–(4) we can add some operations or relations, e.g. $\mathbf{s}$ (the successor), $\leqslant$ or $+$. For many-sorted structures the types of added objects must be determined; e.g., we can add the successor on $\mathbb{N}^{\mathrm{x}}$, and not on $\mathbb{N}^{\mathrm{y}}$.

In every statement about theories of GPT the considered possibility from the above must be specified. Usually (but not always) (1) $\approx$ (2) and (3) $\approx$ (4) from the decidability and definability point of view.

3. An example

It is almost obvious that adding new operations or relations into a structure associated to a GPT can cause that its theory changes from decidable to undecidable. Now we show that the change of number of sorts can have the same effect. We shall give an example of a GPT for which the theory of the associated two-sorted structure is decidable and that of one-sorted one is undecidable. No additional operations or relations will be used. The example is given in Fig. 2.

Theorem 3.1. *Let* $\mathscr{B} = (\{2,0,1\}; *, 2)$, *where*

$$2*0=0, \quad 0*0=1, \quad 1*0=2 \quad \textit{and}$$
$$x*y=x \quad \textit{otherwise (i.e., if } y \neq 0).$$

and let $H = \mathrm{GPT}(\mathscr{B}, 0)$. *Then*

(i) *The addition and multiplication on* $\mathbb{N}$ *are first order definable in the structure* $\langle \mathbb{N}; H\rangle$.

(ii) *The elementary theory of* $\langle \mathbb{N}; H\rangle$ *is undecidable.*

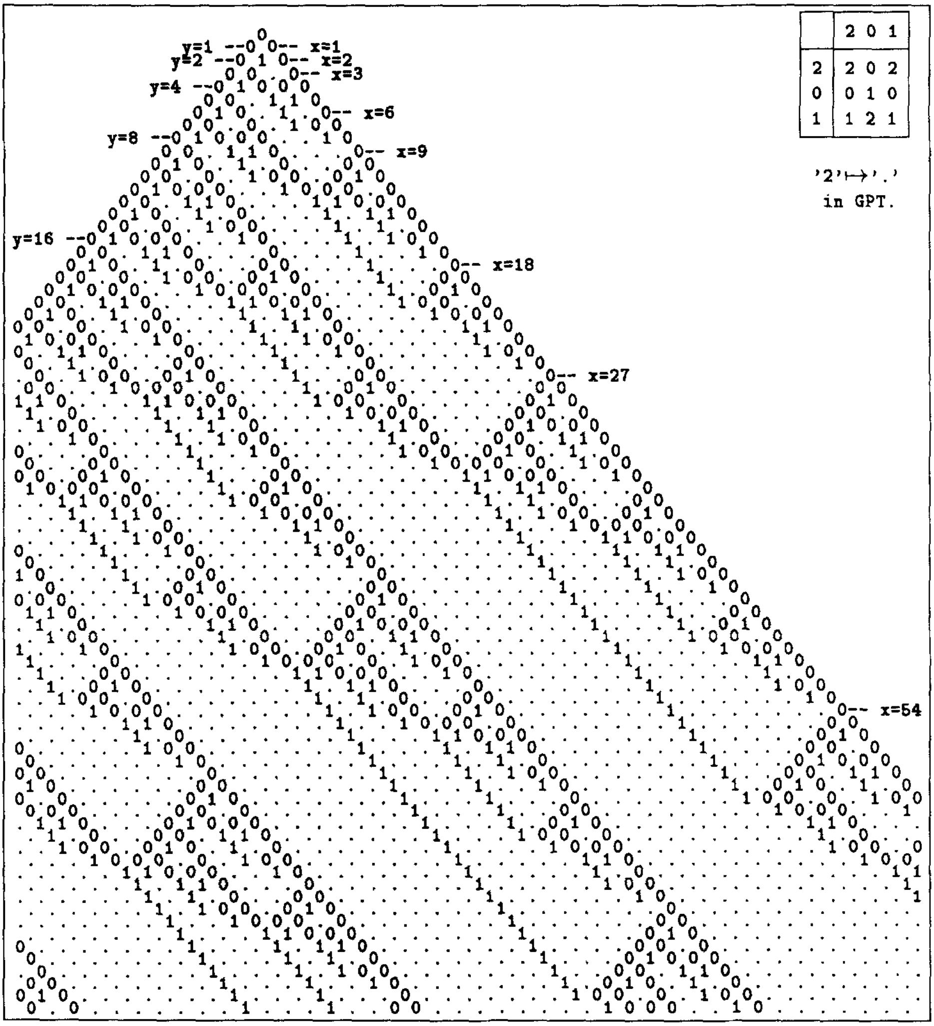

Fig. 2.

The same holds also for the structure $\langle \mathbb{N}; \mathrm{EqH} \rangle$, *where*

$$\mathrm{EqH} = \{(x,y,z,w) \in \mathbb{N}^4 \mid H(x,y) = H(z,w)\}.$$

However, the situation for many-sorted structures introduced above is quite different:

Theorem 3.2. *Let* $\mathscr{B} = (\{2,0,1\}; *, 2)$, $H = \mathrm{GPT}(\mathscr{B}, 0)$ *and* EqH *be as in Theorem* 3.1. *Then the elementary theory of* $\langle \mathbb{N}^{\mathrm{x}}, \mathbb{N}^{\mathrm{y}}, \{0,1,2\}; H \rangle$ *is decidable.*

The same holds also for the structure $\langle \mathbb{N}^{\mathrm{x}}, \mathbb{N}^{\mathrm{y}}; \mathrm{EqH} \rangle$ *where* EqH *is considered as a subset of* $\mathbb{N}^{\mathrm{x}} \times \mathbb{N}^{\mathrm{y}} \times \mathbb{N}^{\mathrm{x}} \times \mathbb{N}^{\mathrm{y}}$.

Instead of proofs we only explain the substantial difference between the situation in Theorem 3.1 and Theorem 3.2. (Both theorems are proved in [11], the first one also in [8].) Let $x \sqsubseteq_k y$ mean that the k-ary digits of x are less than or equal to the corresponding k-ary digits of y. In the structures from Theorem 3.1 both $\sqsubseteq_2$ and $\sqsubseteq_3$ on $\mathbb{N}$ are definable; we can use them to define $+$, $\times$. In the structures from Theorem 3.2 we can define $\sqsubseteq_2$ on $\mathbb{N}^{\mathrm{y}}$ and $\sqsubseteq_3$ on $\mathbb{N}^{\mathrm{x}}$, but these relations remain separated.

Theorem 3.2 can be proved by reduction to the WMSO theory of successor (i.e., that of $\langle \mathbb{N}; \mathbf{s} \rangle$). The elements of $\mathbb{N}^{\mathrm{x}}$ will be coded as ordered pairs of disjoint finite sets (using ternary number system) and the elements of $\mathbb{N}^{\mathrm{y}}$ as finite sets (using binary number system).

Notice that the natural bijection between $\mathbb{N}^{\mathrm{x}}$ and $\mathbb{N}^{\mathrm{y}}$ is not first order definable in the structures from Theorem 3.2. Analogously, the relation

$$\left\{ (X, Y, Z) \,\middle|\, X \cap Y = \emptyset \wedge \sum_{i \in X} 3^i + 2 \cdot \sum_{j \in Y} 3^j = \sum_{k \in Z} 2^k \right\}$$

is not definable in WMSO theory of successor. Informally, although we can speak about the binary and the ternary number system (and define the addition in each of them), we cannot define the natural relationship between them.

4. Decidability results for theories of GPT modulo an integer

In this case decidability or undecidability of the theory of a GPT mainly depends on the factorization of the modulus. The results are summarized in Table 1. Decidability results are obtained by reduction to the weak monadic second order (WMSO) theory of successor.

The addition $+$ is not definable in any structure $\langle \mathbb{N}; B_n \rangle$, n prime. However, it is definable in every such structure if $n > 0$ is composite. Moreover, if $n > 0$ has two distinct prime divisors then $\times$ is definable, too, and hence the elementary theory of $\langle \mathbb{N}; B_n \rangle$ is undecidable (it was proved in [6]).

Table 1

Type of n	Structure	Definable $+$	Definable $\times$	Theory D/U	Nontriv. autom.
n prime	$\langle \mathbb{N}; B_n \rangle$	No	No	D	Yes
	$\langle \mathbb{N}; B_n, \mathbf{s} \rangle$	Yes	No	D	No
	$\langle \mathbb{N}; B_n, \times \rangle$	Yes	Yes	U	No
	$\langle \mathbb{N}; B_n, \mathrm{Sq} \rangle$	Yes	Yes	U	No
n prime power (exponent $\geqslant 2$)	$\langle \mathbb{N}; B_n \rangle$	Yes	(No)	(D)	No
	$\langle \mathbb{N}; B_n, \times \rangle$	Yes	Yes	U	No
	$\langle \mathbb{N}; B_n, \mathrm{Sq} \rangle$	Yes	Yes	U	No
the other $n > 1$	$\langle \mathbb{N}; B_n \rangle$	Yes	Yes	U	No

By [9] addition is definable in the structures $\langle \mathbb{N}; B_n, \mathrm{s}\rangle$, n prime, and by [12] the defining formula can be chosen independently on n. The elementary theory of every structure $\langle \mathbb{N}; B_n, +\rangle$ is decidable for every prime n. In the proof a reduction to WMSO theory of successor is used.

The set of squares Sq and any B_n, $n > 1$, suffice to define both addition and multiplication. Two letters in parentheses in the table denote conjectures; however, they were proved by Bès in [1] (and also presented at Logic Colloquium'94).

The table does not contain special results for $n = 2$. For example, the operations $+$, $\times$ are not WMSO definable in $\langle \mathbb{N}; B_2\rangle$. Nevertheless, the WMSO theory of the structure $\langle \mathbb{N}; B_2\rangle$ is undecidable; the elementary theory of $\langle \mathbb{N}; +, \times\rangle$ can be interpreted in it. The same hold for every prime instead of 2.

5. Decidability results for theories of GPT of small algebras

The results given in Table 2 show by which cardinality of the underlying algebra the theory of a GPT or a class of GPT becomes undecidable. In all considered structures G can be replaced by EqG. The last column corresponds to classes of (structures associated to) GPT, the other columns to single GPT. First order theories are considered except of two last lines (whose concern WMSO theories).

We explain the meaning of symbols in the table:

"U" means "(sometimes) undecidable", i.e.: There is an algebra of specified cardinality such that the (first order) theory of the corresponding structure (or class of structures) is undecidable.

"D" means "(always) decidable", i.e.: For every algebra … the theory … is decidable.

"∀U" means "always undecidable", i.e.: For every algebra … the theory … is undecidable.

Table 2

Structure	$\lvert\mathbf{A}\rvert$	$G = \mathrm{GPT}(\mathscr{A}, w)$ for $\mathscr{A} = (\mathbf{A}; *, \mathrm{o})$,		
		$w \in \mathbf{A}$	$w \in \mathbf{A}^+$	all $w \in \mathbf{A}^+$
$\langle \mathbb{N}; G, +, \times\rangle$	$\geqslant 1$	∀ U (not interesting here)		
$\langle \mathbb{N}; G, \times\rangle$	1	d	d	∀ U
	$\geqslant 2$	U	u	∀ u
$\langle \mathbb{N}; G, +\rangle$	1	d	d	D
	2	D	d	∀ U
	$\geqslant 3$	U	u	∀ u
$\langle \mathbb{N}; G\rangle$	1	d	d	d
	2	d	d	U
	$\geqslant 3$	U	u	u
WMSO theory	1	d	d	(? D)
of $\langle \mathbb{N}; G\rangle$	$\geqslant 2$	U	u	u

Small letters d, u have the same meaning as capitals D, U but they denote less principal results or easy corollaries. (Of course, the classification can be very subjective.) The questionmark denotes a conjecture.

We can see that the considered theories can be undecidable already for $|\mathbf{A}| = 3$; this bound is exact, and does not depend on $|w|$ and the presence of the additional operation s or $+$. It is diminished to 2 if the additional operation is $\times$; the necessary GPT for that is B_2. The same will happen if the additional relation is the divisibility relation or the set of squares as a unary relation. Smaller cardinalities also suffice when the theories of classes of structures are considered. (For $|\mathbf{A}| = 1$ we have only the sets $\mathbb{D}_n$ in essential.) The conjecture for WMSO theories is rather subtle because for the structures $\langle \mathbb{N}; G_1, G_2 \rangle$, where both G_i are GPT, "(? D)" ought to be replaced by "U".

References

[1] A. Bès, On Pascal triangles modulo a prime power, Ann. Pure Appl. Logic 89 (1997) 17–35.

[2] B.A. Bondarenko, Generalized Pascal triangles and pyramids, their fractals, graphs and applications (in Russian), "Fan", Tashkent, 1990.

[3] J.R. Büchi, Weak second-order arithmetic and finite automata, Zeitschrift für Math. Logik Grundlagen Math. 6 (1960) 66–92.

[4] I. Korec, Generalized Pascal triangles, in: K. Halkowska, S. Stawski (Ed.), Proc. V Universal Algebra Symp., Turawa, Poland, May 1988, World Scientific, Singapore, 1989, 198–218.

[5] I. Korec, Definability of arithmetic operations in Pascal triangle modulo an integer divisible by two primes, Grazer Mathematische Berichte 318 (1993) 53–61.

[6] I. Korec, Generalized Pascal triangles, their relation to cellular automata and their elementary theories, J. Dassow, A. Kelemenová (Eds.): Development in theoretical computer science, Proc. 7th IMYCS, Smolenice 92, November 16–20, 1992, Gordon and Breach, London, 1994, pp. 59–70.

[7] I. Korec, Structures related to Pascal's triangle modulo 2 and their elementary theories, Math. Slovaca 44 (5) (1994) 531–554.

[8] I. Korec, Decidable and undecidable theories of generalized Pascal triangles, in: K. Denecke, O. Lüders (Eds.), General Algebra and Discrete Mathematics, Potsdam, Heldermann-Verlag, Berlin, 1995, pp. 169–179.

[9] I. Korec, Elementary theories of structures containing generalized Pascal triangles modulo a prime, in: S. Shtrakov and Iv. Mirchev (Eds.), Discrete Mathematics and Applications, Blagoevgrad/Predel, 1994, Blagoevgrad, 1995, pp. 91–102.

[10] I. Korec, Undecidable elementary theories of classes of generalized Pascal triangles, Tatra Mountains Math. Publ. 5 (1995) 151–168.

[11] I. Korec, Substantially different theories of two structures associated to the same generalized Pascal triangle, in: I. Chajda, F. Halaš, F. Krutský (Eds.), Summer School on General Algebra and Ordered Sets, Horní Lipová, 1994 Palacký University, Olomouc, 1994, pp. 63–78.

[12] I. Korec, Undecidability and uniform definability in classes of structures related to Pascal triangles modulo n, Tatra Mountains Math. Publications, submitted.

[13] H. Lugowski, Grundzüge der Universellen Algebra, Teubner, Leipzig, 1976.

[14] W. Wechler, Universal Algebra for Computer Scientists, Springer, Berlin Heidelberg, 1992.