Neutrality

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(Joint with Roman Kossak)

Notions of genericity

Various notions of genericity have been studied in model theory / models of PA:

- Cohen/Feferman "arithmetic forcing"
- Robinson forcing
- Chatzidakis-Pillay (1998), Dolich-Miller-Steinhorn (2013 and 2016): generic expansions of theories

We base our ideas off of Chatzidakis-Pillay: a subset should be "generic" if it does not affect definability in the structure. Their exact constructions do not work in the context of PA, so we make the following definition (suggested by Dolich):

Definition

Let $M \models \mathrm{PA}$ and $X \subseteq M$. We say X is neutral if, for all $a \in M$, $\mathrm{dcl}^{(M,X)}(a) = \mathrm{dcl}^{M}(a)$.

Neutral sets

Some examples of neutral sets:

- if $M \models \mathsf{PA}$ is a prime model, any subset is neutral
- if $X \subseteq M$ is 0-definable, then X is neutral.
- for any $M \models \mathsf{PA}$, the standard cut ω is neutral (non-trivial, due to Kanovei)

What if we insist that X is undefinable and inductive?

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What if we insist that X is undefinable and inductive?

Lemma

Let $M \models \mathrm{PA}$ and $X \subseteq M$ be inductive. Then X is neutral iff for all $K \prec M$, $(K, X \cap K) \prec (M, X)$.

Proof.

Since X is inductive, we have definable Skolem functions for (M, X). So if $(M, X) \models \exists x \phi(x, a)$ for some $a \in K$, then there is a Skolem term t such that $(M, X) \models \phi(t(a), a)$. By neutrality, $t(a) \in K$. $\square$

Neutrally expandable models

Recall: in a model $M \models \mathrm{PA}$, a subset $X \subseteq M$ is called a class if for all $a \in M$, the set $X \cap [0, a)$ is definable (with parameters) in M. Equivalently, X is a class if for each a, the set $X \cap [0, a)$ has a code.

Definition

$M \models \mathrm{PA}$ is neutrally expandable if it has an undefinable neutral class.

Notes:

- Every prime model of PA is neutrally expandable.
- Suppose $M_0 \models \mathrm{PA}$ is a non-standard prime model. Then if $M_0 \prec_{\mathrm{cof}} M$, M is neutrally expandable. (Proof on the next slide)

Neutrally expandable models

Proposition

Suppose $M_0 \models \mathrm{PA}$ *is a non-standard prime model. Then if* $M_0 \prec_{cof} M$, M *is neutrally expandable.*

Proof.

Let $X_0 \subseteq M_0$ be any inductive, undefinable subset (one can use arithmetic forcing to find one). Then, since X_0 is a class, for each $a \in M_0$, there is a code c_a for the set $X_0 \cap [0, a)$. Define

$$X = \bigcup_{a \in M_0} \{x < a : M \models x \in c_a\}.$$

Then, $(M_0, X_0) \prec (M, X)$ so X is inductive (this construction is due, independently, to Kotlarski and Schmerl). Further, for each $M_0 \preccurlyeq K \preceq M$, we have $(M_0, X_0) \preccurlyeq (K, X \cap K) \preccurlyeq (M, X)$ and so X is neutral. □

Recursively saturated models

Other neutrally expandable models exist. However, recursively saturated models are not neutrally expandable.

Theorem

If N has a recursively saturated elementary submodel, and $X \subseteq N$ is a class, then X is neutral iff X is 0-definable.

Later we will use this result to show that there is no "theory of neutrality":

Theorem

There is no theory T (in $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$) extending PA *such that for any recursively saturated M and any set X, X is neutral iff $(M, X) \models T$.*

Lemma 1

Lemma

Let $M \prec N$ and $X \subseteq N$ a neutral class. Then $X \cap M$ is a class of M.

Proof.

Define the term:

$$m(x) = \min\{y : \forall z < x(z \in y \leftrightarrow z \in X)\}$$

This is well-defined because X is a class: for each x, there is a y which codes the bounded, definable set $X \cap [0, x)$. Since X is neutral, if $a \in M$, then $m(a) = b \in M$ as well, and so $X \cap M$ is a class. □

Conservativity

Recall: N is a conservative elementary extension of M (written $M \prec_{\text{cons}} N$) if for each $X \in \text{Def}(N)$, $X \cap M \in \text{Def}(M)$.

Theorem

Let $X \subseteq N$ be a class. If some $M \prec N$ is a proper conservative elementary extension of its prime submodel, then X is neutral iff X is 0-definable.

This follows from Lemma 1: if X is neutral, then $X \cap M$ is a class. Letting M_0 be the prime submodel, then $X \cap M_0$ is 0-definable using a formula $\phi(x)$. One can easily show that $\phi(x)$ defines X in N.

Proof of Main Theorem

Theorem

If N has a recursively saturated elementary submodel, and $X \subseteq N$ is a class, then X is neutral iff X is 0-definable.

If N has a recursively saturated submodel, then it has elementary submodels which are conservative extensions of its prime submodel. To see this, just let $p(x)$ be any (recursive) definable type, and let $a \in N$ realize p. Then,

$$\mathrm{Scl}(0) \prec_{\mathrm{cons}} \mathrm{Scl}(a) \prec N$$

Recall: a type $p(x) \in S_1(T)$ is definable if, for each $\phi(u, x) \in \mathcal{L}$, there is $\sigma(u)$ such that for every closed term t, $\phi(t, x) \in p(x)$ iff $T \vdash \sigma(t)$. By definition, an elementary extension of a model of PA generated by an element realizing a definable type is a conservative extension.

Neutrality is not first order

Let $X \subseteq \mathbb{N}$ be any undefinable set, and $(\mathbb{N}, X) \prec (M, Y)$ some recursively saturated elementary extension. Then Y is an undefinable, inductive subset of M, but cannot be neutral because M is not neutrally expandable.

Recall our second result, following the same theme:

Theorem

There is no theory T (in $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$) extending PA *such that for any recursively saturated M and any set X, X is neutral iff $(M, X) \models T$.*

Proof

Suppose T is such a theory. If M is recursively saturated, then for every $X \subseteq M$,

$$X \text{ is 0-definable} \Leftrightarrow (M, X) \models T + \text{“}X \text{ is a class''}.$$

Let M be countable and recursively saturated such that $T \in \mathrm{SSy}(M)$. Let S be the theory

$$T + \text{“}X \text{ is a class''} + \{\exists x(\neg x \in X \Leftrightarrow \phi(x)) : \phi \in \mathcal{L}_{\mathrm{PA}}\}.$$

Then $S \in \mathrm{SSy}(M)$, and by resplendency M has an expansion $(M, Y) \models S$. So Y is a neutral class, but is not 0-definable.

Local Neutrality

Definition

Let $M \models \mathrm{PA}$ and $A \subseteq M$. We say $X \subseteq M$ is *A-neutral* if, for all $a, b \in A$, $a \in \mathrm{dcl}^{(M,X)}(b)$ if and only if $a \in \mathrm{dcl}^{M}(b)$.

We have the following results, whose proofs are due to Jim Schmerl:

Theorem

Let M be a countable recursively saturated model of PA*, and $A \subseteq M$ a bounded subset of M. Then M has an inductive, undefinable A-neutral subset X.*

Theorem

Let M be a countable recursively saturated model of PA*. There are $A \subseteq_{cof} M$ and an undefinable inductive $X \subseteq M$ that is A-neutral.*

Proof 1

Theorem

Let M be a countable recursively saturated model of PA, *and $A \subseteq M$ a bounded subset of M. Then M has an inductive, undefinable A-neutral subset X.*

We assume $A \prec_{\text{end}} M$. (If not, find the smallest elementary cut of M containing A.) Let $p(x)$ be a minimal type realized in M, and $C \subseteq M \setminus A$ a cofinal set of realizations of p. Let $N = \text{dcl}(A \cup C)$, and so $A \prec_{\text{cons}} N \prec_{\text{cof}} M$.

Notice: if $Y \subseteq N$ is definable using parameters $a \in A$ and $\bar{c} \in C$, then $Y \cap A$ is definable in A using only parameter a (by the definability of $p(x)$).

Proof 1

Let $G \subseteq \text{dcl}(C)$ be generic (in the sense of arithmetic forcing). There is $X \subseteq M$ such that $(\text{dcl}(C), G) \prec (M, X)$; such X is also generic. Then if $a, b \in A$ are such that $a \in \text{dcl}^{(M,X)}(b)$, that means

$$(M, X) \models \phi(a, b) \wedge (\exists! x \phi(x, b)).$$

Hence, by the forcing lemma for arithmetic, there is $p \in X$ such that $M \models p \Vdash \phi(a, b) \wedge (\exists! x \phi(x, b))$. Let $q \in G$ extend p, and let

$$Y = \{\langle x, y \rangle : N \models q \Vdash [\phi(x, y) \wedge \exists! z \phi(z, y)]\}$$

Since Y is definable using only $q \in G$, and $q \in \text{dcl}(C)$, $Y \cap A$ is 0-definable in A. Hence $a \in \text{dcl}(b)$.

Proof 2

Question: can you be neutral with respect to an unbounded set? Easy answer:

Theorem

Let M be a countable recursively saturated model of PA*. There are $A \subseteq_{cof} M$ and an undefinable inductive $X \subseteq M$ that is A-neutral.*

Let X be undefinable, inductive such that (M, X) is recursively saturated (find such an X using chronic resplendence). Let $p(x)$ be an unbounded type in the language $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$ realized in (M, X). Then let A be the set of realizations of p.

- A is unbounded in M, since p is an unbounded type
- For any a, b, if $a \in \mathrm{dcl}^{(M,X)}(b)$, then $\mathrm{tp}^{(M,X)}(a) \neq \mathrm{tp}^{(M,X)}(b)$ by Ehrenfeucht's Lemma. But if $a, b \in A$, they must realize the same type, and hence $a \in \mathrm{dcl}^{(M,X)}(b)$ if and only if $a = b$.

Some open questions

- Classify all neutrally expandable models of PA.
- How much of PA is needed for these proofs? ie, what can we say about neutrally expandable models of $I\Delta_0 + \exp$?
- Classify those subsets A of a recursively saturated model for which there exist undefinable A-neutral classes.

Thank you!

Questions?