Modest automorphisms of Presburger arithmetic

Simon Heller

May 28, 2019

We consider Presburger arithmetic, the theory of the integers as a discrete ordered abelian group with addition in the language:

$\mathcal{L}_{Pr} = \{+, -, <, 0, 1, P_n (n = 2, 3, \ldots)\}$

where we interpret P_n as the unary predicate for the elements of $\mathcal{M}$ divisible by n.

This work builds on the work of Harnik (1986) and Llewellyn-Jones (2001).

Three basic definitions:

1. The *divisibility type* $\rho(a)$ of $a \in M$ is the sequence $(r_2, r_3, \ldots, r_n, \ldots)$ where r_n is the residue of a mod n.

2. $a, b \in M$ are in the same *magnitude class* iff there are positive integers m, n such that $m|a| < |b| < n|a|$.

3. for positive a, b in the same magnitude class, the *standard part* of a over b $\mathrm{st}(\frac{a}{b})$ is

$$\sup\{q | qb < a, q \in \mathbb{Q}^+\}$$

and for $0 < a < b$ in distinct magnitude classes,

$$\mathrm{st}\left(\frac{a}{b}\right) = 0 \text{ and } \mathrm{st}\left(\frac{b}{a}\right) = \infty.$$

The standard part definition extends to non-positive elements of M:
For a, b in the same magnitude class and $a < 0 < b$:

$$\mathrm{st}\left(\frac{a}{b}\right) = -\mathrm{st}\left(\frac{|a|}{b}\right).$$

We define automorphisms g of $\mathcal{M}$ to be *modest* if

$$\mathrm{st}\left(\frac{g(a)}{a}\right) = 1.$$

All the definitions above transfer in a natural way to the quotient structure $\mathcal{M}/\mathbb{Z}$. Note that:

1. $\mathcal{M}/\mathbb{Z}$ is a divisible abelian group

2. Each element of $\mathcal{M}/\mathbb{Z}$ inherits a divisibility type from $\mathcal{M}$, called a *color*), and we consider the quotient structure with unary predicates for these colors.

The results below are for recursively saturated models . Such models satisfy the following:

1. the colors are dense in $\mathcal{M}/\mathbb{Z}$;
2. for $x, y, z \in \mathcal{M}/\mathbb{Z}$ with $z \neq 0$, there is some $w \neq 0$ for which $\mathrm{st}(w/z) = \mathrm{st}(x/y)$; and
3. the set of magnitude classes in $\mathcal{M}/\mathbb{Z}$ is a dense linear order with respect to the ordering $<$ with least element 0 and no greatest element.

These features of recursively saturated models of Presburger arithmetic were identified and studied by Harnik (1986) and Llewellyn-Jones (2001).

Using a back-and-forth method, we can construct a modest automorphism σ of a countable pseudo-recursively saturated model of Presburger arithmetic that satisfies the following properties:

(1) the fixed point set F of σ is a convex, dense set of magnitude classes containing the standard integers;
(2) σ is strictly increasing on the positive part of $M \setminus F$ (and strictly decreasing on the negative part of $M \setminus F$); and
(3) the $\mathbb{Z}$-chains containing elements of the set of *differences* $D = \{x | \exists w(\sigma(w) - w = x)\}$ are dense and co-dense in the $\mathbb{Z}$-chains in F.

The automorphism σ corresponds to an automorphism of the quotient structure $\mathcal{M}/\mathbb{Z}$ with the analogous properties, except that the set of differences is now dense and codense in the fixed-point set.

By varying the method used to construct σ, we can also show that there is a maximal modest automorphism τ, with the following properties:
(1) the fixed point set F of τ is $\mathbb{Z}$;
(2) τ is modest and strictly increasing on the positive part of M; and
(3) the set of $\mathbb{Z}$-chains containing an element of the set of differences $D = \{x | \exists w(\tau(w) - w = x\}$ is dense in the $\mathbb{Z}$-chains in M.

By adding a predicate for the set of automorphic differences D, we are able to give a recursive axiomatization T^* of the quotient structure $\mathcal{M}/\mathbb{Z}$, expanded by σ, and prove the following:

Theorem. Let $\mathcal{M}^* \models T^*$, and let $\phi(x, \bar{y})$ be a quantifier-free formula that is a conjunction of literals in the expanded language. Then there is a quantifier-free formula θ such that $\mathcal{M}^* \models \exists\phi(x, \bar{y}) \leftrightarrow \theta(\bar{y})$.

Corollary. The axiomatization T^* is complete.

We can similarly prove quantifier elimination and give and axiomatizion T for the Presburger structure expanded by σ; in this case we need to also add predicates for elements being in the same $\mathbb{Z}$-chain as a difference (either above or below the difference), and for two elements being in the same $\mathbb{Z}$-chain.

Definable sets and algebraic closure

In the quotient $(\mathcal{M}/\mathbb{Z}, \sigma)$

(1) the definable sets are unions of convex sets and infinite sets dense in convex sets

and

(2) the algebraic closure of a set $A = \{a_1, \ldots, a_n\} \in M/\mathbb{Z}$ is the set of all $\mathbb{Q}$-linear combinations of the elements of A and their associated differences $\{\sigma(a_1) - a_1, \ldots, \sigma(a_n) - a_n\}$

In the Presburger structure expanded by σ,

(1) the definable sets are unions of convex sets, convex sets intersected with periodic sets, cosets of the set of σ-differences, and convex sets with cosets of the set of σ-differences removed

(2) the algebraic closure of $A = \{a_1, \ldots, a_n\} \subset M$ is the set of $\mathbb{Z}$-linear combinations of the elements of A and their associated differences $\{\sigma(a_1) - a_1, \ldots, \sigma(a_n) - a_n\}$, and quotients of such linear combinations by $k \in \mathbb{Z}$ if the linear combination is divisible by k.

Using the classification of the definable sets in both the quotient structure and the Presburger structure, we can show:
Theorem. $(\mathcal{M}/\mathbb{Z}, \sigma)$ and $(\mathcal{M}, \sigma)$ both have DP rank 2.

Open questions:
(1) Can we find other automorphisms of countable, recursively saturated models of Presburger arithmetic such that when we expand by the automorphisms, we can prove quantifier elimination and axiomatizability?
(2) Given $n > 2$ finite, can we find an automorphism such that expansion by that automorphism has DP rank n?