# Necessarily non-analytic induction proofs

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# A non-analytic induction proof

Define $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is the sum of the first $n$ odd natural numbers:

$$\begin{aligned} f(0) &:= 0, \\ f(n+1) &:= f(n) + 2n + 1. \end{aligned}$$

That is, we have

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1, \\ f(2) &= 1 + 3, \\ f(3) &= 1 + 3 + 5, \\ f(4) &= 1 + 3 + 5 + 7, \\ &\vdots \end{aligned}$$

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**Fact**

*$f(n)$ is a perfect square for all $n$: For all natural numbers $n$ there is a natural number $m$ such that $f(n) = m^2$.*

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Let us try to prove this by "straightforward induction"; that is, let us try to prove the following.

- Base case: $f(0)$ is a perfect square.
- Induction step: For all natural numbers $n$, if $f(n)$ is a perfect square then $f(n+1)$ is a perfect square.

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- We want to prove that $f(n+1) = m^2$ for some natural number $m$.
- We have

$$\begin{aligned} f(n+1) &= f(n) + 2n + 1 && \text{(by definition)} \\ &= k^2 + 2n + 1 && \text{(by induction hypothesis)} \end{aligned}$$

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but $k^2 + 2n + 1$ is not a perfect square for arbitrary natural numbers $k$ and $n$ so how do we proceed from here?

# A non-analytic induction proof

Let us try a different approach. Our fact follows immediately from the following stronger fact.

**Fact**

$f(n) = n^2$ *for all natural numbers* $n$.

(This fact is stronger in the sense that it logically implies the previous fact, while the previous fact does not logically imply this fact.)

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**Fact**

*$f(n) = n^2$ for all natural numbers $n$.*

(This fact is stronger in the sense that it logically implies the previous fact, while the previous fact does not logically imply this fact.)

Let us try to prove this fact by "straightforward induction"; that is, let us try to prove the following.

- Base case: $f(0) = 0^2$.
- Induction step: For all natural numbers $n$, if $f(n) = n^2$ then $f(n+1) = (n+1)^2$.

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$$\dfrac{\begin{array}{c}\vdots \\ \varphi(0)\end{array} \qquad \begin{array}{c}\vdots \\ \forall x\colon\ \varphi(x) \to \varphi(x+1)\end{array}}{\forall x.\, \varphi(x)} \quad .$$

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- The typical form of a "proof by a strengthening of the induction hypothesis":

$$\begin{matrix}\dfrac{\begin{matrix}\vdots \\ \psi(0)\end{matrix} \qquad \begin{matrix}\vdots \\ \forall x\colon\ \psi(x) \to \psi(x+1)\end{matrix}}{\forall x.\, \psi(x)} \\ \vdots \\ \forall x.\, \varphi(x)\end{matrix} \quad .$$

# Terminology

There need not always be any precise sense in which $\psi(x)$ is stronger than $\varphi(x)$. Thus, following Hetzl and Wong, we use the more general terminology "non-analytic induction proofs".[1]

[1] Stefan Hetzl and Tin Lok Wong (2018): "Some observations on the logical foundations of inductive theorem proving".

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- Question: Take a non-analytic induction proof (for example, the proof we just saw). Is the non-analyticity of this proof necessary?
- It is not immediately obvious how to make precise sense of this question. For example, if we would use the previously given forms to distinguish analytic induction proofs from non-analytic induction proofs, then any proof of $\forall x.\, \varphi(x)$ could be turned into an analytic induction proof:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall x.\, \varphi(x)\end{array}}{\varphi(0)} \qquad \begin{array}{c}\vdots\\ \forall x.\, \varphi(x)\\ \vdots\\ \forall x\colon\ \varphi(x) \to \varphi(x+1)\end{array}}{\forall x.\, \varphi(x)}\ .$$

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- Hetzl and Wong have made precise nontrivial sense of the notion of "necessarily non-analytic induction proof".
- Our main result so far: Using Hetzl's and Wong's formulation, there is a precise sense in which we must use non-analytic induction to prove "the sum of any initial segment of the odd natural numbers is a perfect square".

# Definitions

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- A first-order language is a *(first-order) language of arithmetic* if and only if it is an $\mathcal{L}_{\text{min}}$-expansion.

**Definition**

Let $L$ be a language of arithmetic and let $\varphi(x)$ be an $L$-formula in the free variable $x$. The *induction instance* for $\varphi(x)$ is the $L$-sentence

$$\mathrm{IND}(\varphi) :\equiv \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x.\, \varphi(x).$$

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Let $L$ be a language of arithmetic. Let $T$ be an $L$-theory. Let $\varphi(x)$ be an $L$-formula in the free variable $x$. $T$ *proves* $\forall x.\, \varphi(x)$ *by necessarily non-analytic induction* if and only if there is an $L$-formula $\psi(x)$ in the free variable $x$ such that

$$
\begin{aligned}
&(1) && T, \mathrm{IND}(\varphi) \nvdash \forall x.\, \varphi(x), \\
&(2) && T \vdash \varphi(0), \\
&(3) && T \vdash \psi(0), \\
&(4) && T \vdash \forall x\colon\ \psi(x) \to \psi(x+1), \\
&(5) && T \vdash \forall x.\, \psi(x) \to \forall x.\, \varphi(x).
\end{aligned}
$$

Under conditions (1)–(5), we also say that $\psi(x)$ *witnesses* that $T$ proves $\forall x.\, \varphi(x)$ by necessarily non-analytic induction.

# Definitions

Let $\mathcal{L}^{\mathsf{OR}}$ be the language of ordered rings—signature $\langle 0, 1, +, \cdot, < \rangle$. We find it very reasonable that working mathematicians take the axioms of the $\mathcal{L}^{\mathsf{OR}}$-theory $\mathsf{PA}^-$—*the theory of the non-negative parts of nontrivial discretely ordered commutative rings*[2]—for granted when doing arithmetic.

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The axioms of $\mathrm{PA}^-$ are:

- associativity of addition: $(x + y) + z = x + (y + z)$,

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- commutativity of addition: $x + y = y + x$,
- commutativity of multiplication: $x \cdot y = y \cdot x$,
- distributivity of multiplication over addition: $x \cdot (y + z) = x \cdot y + x \cdot z$,

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- the order is discrete: $0 < x \to x = 1 \vee 1 < x$,
- 0 is the least element: $x = 0 \vee 0 < x$.

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- Define $L$-formulas $\varphi(x)$ and $\psi(x)$ corresponding to the analytic and non-analytic induction hypotheses, respectively:

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*$\psi(x)$ witnesses that $T$ proves $\forall x.\, \varphi(x)$ by necessarily non-analytic induction.*

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- To show condition (1),

$$T, \mathrm{IND}(\varphi) \nvdash \forall x.\, \varphi(x),$$

we exhibit a non-standard $L$-model $M \models T$ with a non-standard number $c$ such that

$$M \models \varphi(c),$$
$$M \not\models \varphi(c+1).$$

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- $\mathbb{Z}[X] \coloneqq \langle \mathbb{Z}[X], 0, 1, +, \cdot, < \rangle$ is the ordered ring of polynomials in the indeterminate $X$ with coefficients in $\mathbb{Z}$.

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- $\mathbb{Z}[X] := \langle \mathbb{Z}[X], 0, 1, +, \cdot, < \rangle$ is the ordered ring of polynomials in the indeterminate $X$ with coefficients in $\mathbb{Z}$.
- Elements of $\mathbb{Z}[X]$ are polynomials

$$z_n X^n + \cdots + z_1 X^1 + z_0$$

  with $z_0, \ldots, z_n$ in $\mathbb{Z}$ and if $n \neq 0$ then $z_n \neq 0$. $z_n$ is the *leading coefficient* of the polynomial. $n$ is the *degree* of the polynomial.

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- Addition, multiplication and subtraction in $\mathbb{Z}[X]$ are as expected.
- The order can be thought of as taking $X$ to be infinitely large and taking $X^{n+1}$ to be infinitely larger than $X^n$ for each natural number $n$. Making this precise, we may define the order by the clauses

$$\begin{aligned} z_n X^n + \cdots + z_1 X^1 + z_0 > 0 \text{ if and only if } z_n > 0, \\ p > q \text{ if and only if } p - q > 0. \end{aligned}$$

# Proof of our result

- The polynomials in $\mathbb{Z}[X]$ can be divided into the *constant* polynomials
$$z \quad (z \text{ in } \mathbb{Z})$$
and the *non-constant* polynomials
$$pX + z \quad (p \text{ in } \mathbb{Z}[X],\ p \neq 0,\ z \text{ in } \mathbb{Z}).$$

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$$pX + z \quad (p \text{ in } \mathbb{Z}[X],\ p \neq 0,\ z \text{ in } \mathbb{Z}).$$
- Every polynomial in $\mathbb{Z}[X]$ can be uniquely written on one of the above forms.

# Proof of our result

- Let $\mathbb{Z}[X]^+$ be the non-negative part of $\mathbb{Z}[X]$; that is, $\mathbb{Z}[X]^+$ is the substructure of $\mathbb{Z}[X]$ that consists of polynomials of the form

$$z_n X^n + \cdots + z_1 X^1 + z_0$$

with $z_n \geq 0$ (and $z_n = 0$ only if $n = 0$).

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**Fact**

*An $\mathcal{L}^{OR}$-model $M$ is a model of $PA^-$ if and only if $M$ is the non-negative part of a nontrivial discretely ordered commutative ring.*

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**Proof.**

See for example Kaye's *Models of Peano Arithmetic*. $\square$

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$\mathbb{Z}[X]^+ \models PA^-$.

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$\mathbb{Z}[X]^+ \models PA^-$.

**Proof.**

$\mathbb{Z}[X]^+$ is the non-negative part of the nontrivial discretely ordered commutative ring $\mathbb{Z}[X]$. $\square$

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- We want to expand $\mathbb{Z}[X]^+$ to an $L$-model $M \models T$ such that $M \models \varphi(p)$ and $M \not\models \varphi(p+1)$ for some polynomial $p$ in $\mathbb{Z}[X]^+$.

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- To expand $\mathbb{Z}[X]^+$ to an $L$-model $M$ we need to provide an interpretation $f^M : \mathbb{Z}[X]^+ \to \mathbb{Z}[X]^+$ of $f$.

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- To expand $\mathbb{Z}[X]^+$ to an $L$-model $M$ we need to provide an interpretation $f^M : \mathbb{Z}[X]^+ \to \mathbb{Z}[X]^+$ of $f$.
- Recall that

$$T = \mathrm{PA}^- \cup \{f(0) = 0,\ \forall x.\, f(x+1) = f(x) + 2x + 1\}.$$

and that

$$\varphi(x) :\equiv \exists y.\, f(x) = y^2.$$

# Proof of our result

- We want to expand $\mathbb{Z}[X]^+$ to an $L$-model $M \models T$ such that $M \models \varphi(p)$ and $M \not\models \varphi(p+1)$ for some polynomial $p$ in $\mathbb{Z}[X]^+$.
- To expand $\mathbb{Z}[X]^+$ to an $L$-model $M$ we need to provide an interpretation $f^M : \mathbb{Z}[X]^+ \to \mathbb{Z}[X]^+$ of $f$.
- Recall that

  $$T = \mathrm{PA}^- \cup \{f(0) = 0,\ \forall x.\, f(x+1) = f(x) + 2x + 1\}.$$

  and that

  $$\varphi(x) :\equiv \exists y.\, f(x) = y^2.$$

- Thus $f^M$ needs to satisfy the defining equations for $f$ and be such that for some polynomial $p$ in $\mathbb{Z}[X]^+$ we have that $f^M(p)$ is a perfect square in $\mathbb{Z}[X]^+$ while $f^M(p+1)$ is not.

# Proof of our result

- Since $M$ must model (the universal closure of) the recursive defining equation,

$$f(x+1) = f(x) + 2x + 1,$$

we get that $f^M$ must satisfy

$$\begin{aligned} f^M(p) &= f^M((p-1)+1) \\ &= f^M(p-1) + 2(p-1) + 1. \\ &= f^M(p-1) + 2p - 1. \end{aligned}$$

Thus $f^M$ must satisfy

$$f^M(p-1) = f^M(p) - 2p + 1.$$

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- $f^M$ must thus satisfy the equations

$$\begin{aligned} f^M(0) &= 0, \\ f^M(p+1) &= f^M(p) + 2p + 1, \\ f^M(p-1) &= f^M(p) - 2p + 1. \end{aligned}$$

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- The first two equations fixes $f^M$ on the constant polynomials.
- Let $pX + z$ be a non-constant polynomial in $\mathbb{Z}[X]^+$ and let $q$ be any polynomial in $\mathbb{Z}[X]^+$. By the last two equations, setting $f^M(pX + z) = q$ fixes $f^M$ on all polynomials of the form $pX + z'$ ($z'$ in $\mathbb{Z}$); that is, it fixes $f^M$ on $pX + z$ and on the polynomials

$$pX + z + 1,\ pX + z + 2,\ \ldots$$
$$pX + z - 1,\ pX + z - 2,\ \ldots$$

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- In doing so, making sure that
  - $f^M(p)$ is in $\mathbb{Z}[X]^+$ for all (non-constant) $p$ in $\mathbb{Z}[X]^+$,
  - for some (non-constant) $p$ in $\mathbb{Z}[X]^+$, $f^M(p)$ is a perfect square while $f^M(p+1)$ is not.

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  - for some (non-constant) $p$ in $\mathbb{Z}[X]^+$, $f^M(p)$ is a perfect square while $f^M(p+1)$ is not.
- For each $p > 0$ in $Z[X]^+$, define

$$f^M(pX - 1) := pX^2.$$

# Proof of our result

- $f^M(p)$ is in $\mathbb{Z}[X]^+$ for all (non-constant) $p$ in $\mathbb{Z}[X]^+$: We need to worry about the equation

$$f^M(p-1) = f^M(p) - 2p + 1.$$

By construction, $f^M(p)$ is always positive and of greater degree than $p$ for non-constant polynomials $p$. Thus the right hand side will never be negative.

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$$f^M(p-1) = f^M(p) - 2p + 1.$$
By construction, $f^M(p)$ is always positive and of greater degree than $p$ for non-constant polynomials $p$. Thus the right hand side will never be negative.
- We have
$$f^M(X-1) = X^2$$
and
$$\begin{aligned} f^M(X) &= f(X-1) + 2(X-1) + 1 \\ &= X^2 + 2X - 1. \end{aligned}$$
Thus $f^M(X-1)$ is a perfect square in $\mathbb{Z}[X]^+$ while $f^M(X)$ is not. This completes the proof.

# Summary of our result

Let

$$T := \mathrm{PA}^- \cup \{f(0) = 0,\ \forall x.\, f(x+1) = f(x) + 2x + 1\}.$$

and let

$$\varphi(x) :\equiv \exists y.\, f(x) = y^2,$$
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**Fact**

*$\psi(x)$ witnesses that $T$ proves $\forall x.\, \varphi(x)$ by necessarily non-analytic induction.*

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$$T, \mathrm{IND}(\varphi) \nvdash \forall x.\, \varphi(x),$$

we exhibit a non-standard $L$-model $M \vDash T$ with a non-standard natural number $c$ such that $M \vDash \varphi(c)$ and $M \nvDash \varphi(c+1)$.

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- $\mathbb{Z}[X]^+$ is a model of $\mathrm{PA}^-$. We expand $\mathbb{Z}[X]^+$ to an $L$-model $M$ by interpreting $f$ on $\mathbb{Z}[X]^+$.

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  we exhibit a non-standard $L$-model $M \models T$ with a non-standard natural number $c$ such that $M \models \varphi(c)$ and $M \not\models \varphi(c+1)$.
- $\mathbb{Z}[X]^+$ is a model of $\mathrm{PA}^-$. We expand $\mathbb{Z}[X]^+$ to an $L$-model $M$ by interpreting $f$ on $\mathbb{Z}[X]^+$.
- We define our interpretation $f^M$ such that it satisfies the defining equations for $f$ and such that $f^M(X-1)$ is a perfect square in $\mathbb{Z}[X]^+$ while $f^M(X)$ is not. $\square$

# Ideas for future work

Our proof breaks down if we add any sentence to $T$ that is false in $\mathbb{Z}[X]^+$. A natural such sentence that is true in the standard model is "all numbers are even or odd", that is

$$\sigma :\equiv \forall x \exists y.\, x = y + y \vee x = y + y + 1.$$

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Our proof breaks down if we add any sentence to $T$ that is false in $\mathbb{Z}[X]^+$. A natural such sentence that is true in the standard model is "all numbers are even or odd", that is

$$\sigma :\equiv \forall x \exists y.\, x = y + y \vee x = y + y + 1.$$

**Conjecture**

*$\psi(x)$ witnesses that $T \cup \{\sigma\}$ proves $\forall x.\, \varphi(x)$ by necessarily non-analytic induction.*

# Ideas for future work

- Develop more general methods to settle conjectures about necessary non-analyticity (as opposed to the method of hand-crafting countermodels for each particular case).

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- Develop more general methods to settle conjectures about necessary non-analyticity (as opposed to the method of hand-crafting countermodels for each particular case).
- Consider other settings than arithmetic. For example, in computer science, many basic facts of functions on inductive structures seem to require non-analytic induction proofs.

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- Consider other settings than arithmetic. For example, in computer science, many basic facts of functions on inductive structures seem to require non-analytic induction proofs.
- Consider the problem of non-analytic induction proofs from the more proof-theoretical side. Dag Prawitz's recent "The concepts of proof and ground" might be useful.[3]

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# Thanks!

Thanks for listening!