End extensions of models of fragments of PA

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$I\Sigma_n$: induction for Σ_n formulas (plus base theory)

$B\Sigma_n$: $I\Delta_0$ + collection for Σ_n formulas

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$B\Sigma_n$: $I\Delta_0$ + collection for Σ_n formulas

Theorem (MacDowell-Specker, 1961)

Every model of PA has a proper elementary end extension.

J. B. Paris and L. A. S. Kirby. Σ_n-collection schemas in arithmetic, in *Logic Colloquium '77*, 199–029, North-Holland, 1978.

Theorem. For any $n \geq 2$, if M is a countable model of $B\Sigma_n$, then M has a proper Σ_n-elementary end extension satisfying $I\Delta_0$.

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P. Clote. A note on the MacDowell-Specker theorem. *Fund. Math.* 127 (1986), 163–170.

The Kirby-Paris construction used very strongly the countability of the model. In view of the cardinality-free statement of the MacDowell-Specker Theorem, we might expect the conclusion of Theorem 1 to hold for models of any cardinality. Such a possibility was first suggested by A. Wilkie.

<u>Theorem.</u> For any $n \geq 2$, if M is a model of $I\Sigma_n$, then M has a proper Σ_n-elementary end extension satisfying $I\Delta_0$.

Remark. Proofs of the Paris-Kirby and Clote results based on restricted ultrapower constructions

P. Clote and J. Krajíček. Open problems. *Oxford Logic Guides*, volume 23, *Arithmetic, proof theory and computational complexity* (Prague, 1991). Oxford University Press, New York, 1993.

Problem 1 (Fundamental problem F). Does every countable model of $B\Sigma_1$ have a proper end extension satisfying $I\Delta_0$?

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Problem 1 (Fundamental problem F). Does every countable model of $B\Sigma_1$ have a proper end extension satisfying $I\Delta_0$?

A. J. Wilkie and J. B. Paris. On the existence of end extensions of models of bounded induction. In *Logic, methodology and philosophy of science, VIII (Moscow, 1987)*, volume 126 of *Stud. Logic Found. Math.*, 143–161, North-Holland, 1989.

$I\Delta_0$-fullness: saturation condition

Theorem.

For every countable model M of $B\Sigma_1$, if M is $I\Delta_0$-full, then there exists K such that $M \subset_e K$ and K satisfies $I\Delta_0$.

5 natural conditions, each of which implies $I\Delta_0$-fullness
the most natural one: *exp*

REMARK. A direct proof that any countable model of $B\Sigma_1$ *which is closed under exponentiation has a proper end extension to a model of* $I\Delta_0$ *may be obtained by mimicking the proof of Theorem 4 but with "Semantic Tableaux consistency of* Γ*" in place of "*Γ*-full" and adding a new constant symbol* $\pi > M$ *to ensure that the end extension is proper.*

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2016 paper: elaboration of this idea, also for 3 more of the Wilkie-Paris conditions (the 5th is irrelevant)

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2020 paper: application of the same basic idea, to give an alternative proof of Clote's theorem and to prove an extra result

Problem 2. Does every model of $I\Sigma_1$ have a proper end extension satisfying $I\Delta_0$? (recall that $I\Sigma_1 \Rightarrow B\Sigma_1$)

Theorem. Problem 2 has a positive solution, i.e., every model of $I\Sigma_1$ has a proper end extension to a model of $I\Delta_0$.

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Our approach (for both proofs) combines
(a) the well-known procedure of extending a consistent theory to a maximal consistent one
(b) the consideration of structures whose universes are sets of definable elements.

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Theorem. For any $n \geq 1$, $B\Sigma_n \Leftrightarrow L\Delta_n \Rightarrow I\Delta_n$
(see page 63 in P. Hájek and P. Pudlák. *Metamathematics of first-order arithmetic*. Springer, 1993)

Problem 3 (Technical problem no. 34). For $n \geq 1$, is $I\Delta_n$ equivalent to $B\Sigma_n$?

T. Slaman. Σ_n-bounding and Δ_n-induction. *Proc. Amer. Math. Soc.* 132 (2004), 2449–2456.

Theorem. (a) For $n \geq 2$, $I\Delta_n \Leftrightarrow B\Sigma_n$.
(b) $I\Delta_1 + exp \Rightarrow B\Sigma_1$ (hence $I\Delta_1 + exp \Leftrightarrow B\Sigma_1 + exp$).

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Problem 4. Does every model of $B\Sigma_1 + exp$ have a proper end extension satisfying $I\Delta_0$?

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Problem 5. Does every model of $I\Delta_1 + exp$ have a proper end extension satisfying $I\Delta_0$?

Remarks. Without assuming Slaman's result,
(i) If "yes" to Problem 5, then "yes" to Problem 4.
(ii) If "yes" to Problem 5, then (by a well-known result) $I\Delta_1 + exp \Rightarrow B\Sigma_1$, i.e., (b) of Slaman's result follows.

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N. Thapen. A note on Δ_1 induction and Σ_1 collection. *Fund. Math.* 186 (2005), 79–84.

e_p is the axiom $\forall x \exists y\,(x<p(y) \wedge$ "x^y exists"),
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Theorem. $I\Delta_1+e_p \Rightarrow B\Sigma_1$.

Remarks. (i) Thapen's result implies part (b) of Slaman's result, since exp is (equivalent to) e_p for the specific primitive recursive function $p(y)=y+1$.
(ii) Another instance of e_p is Ω_1, i.e., the axiom $\forall x \exists y\,(y=x^{|x|})$, where $|x|$ denotes the length of x.

Problem 6. Does every model of $I\Delta_1+e_p$ have a proper end extension satisfying $I\Delta_0$?

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Remark. $I\Delta_1 + e_p$ is far stronger than *IOpen*, so our method needs a lot of improvement!