Metamathematics of the Global Reflection Principle

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Abstract

The talk is planned as a survey of recent developments concerning the Global Reflection Principle for Peano Arithmetic (GRP), to wit the formal assertion that all theorems of PA are true. We discuss various additional principles that, when considered over a natural axiomatic theory of truth, are equivalent to GRP. The relevant results are presented in [5] and [2].

During the talk we discuss the formal properties of the extension of Peano Arithmetic with a very natural soundness assertion expressing

$$\text{All axioms of PA are true.} \quad (*)$$

One way to capture the meaning of the above is to use the uniform reflection principle (hereafter REF(PA)) i.e. the infinite collection of arithmetical sentences

$$\forall x\big(\mathrm{Prov}_{\mathrm{PA}}(\phi(\dot{x})) \to \phi(x)\big),$$

where $\phi(x)$ is a formula in the language of arithmetic and $\mathrm{Prov}_{\mathrm{PA}}(x)$ is the canonical provability predicate (for PA). Another way is to make use of a primitive truth predicate T and express $(*)$ as a single sentence

$$\forall \phi \in \mathcal{L}_{\mathrm{PA}}\big(\mathrm{Prov}_{\mathrm{PA}}(\phi) \to T(\phi)\big) \qquad (\mathrm{GR}(\mathrm{PA}))$$

To make sense of the above, PA needs to be extended with some axioms characterizing T as a truth predicate (for the language of arithmetic). Arguably the most natural way to do it is to formalize the usual inductive Tarski's truth clauses and add to PA the following statements

CT1 $\forall s, t\big(T(s = t) \equiv s^\circ = t^\circ\big)$.

CT2 $\forall \phi, \psi\big(T(\phi \vee \psi) \equiv T(\phi) \vee T(\psi)\big)$.

CT3 $\forall \phi\big(T(\neg\phi) \equiv \neg T(\phi)\big)$.

CT4 $\forall \phi(v)\big(T(\forall v \phi(v)) \equiv \forall x T(\phi[\underline{x}/v])\big)$.

In the above s, t range over (the Gödel codes of) closed terms, ϕ, ψ range over (the Gödel codes of) sentences and $\phi(v)$ ranges over (the Gödel codes of) formulae with at most one free variable. Finally $\phi[s/v]$ denotes the result of substituting term s for a variable v in a formula ϕ and $\underline{x}$ denotes the canonical numeral naming x.

Definition 1. $\mathrm{CT}^-[\mathrm{PA}]$ is the extension of PA with the axioms CT1, CT2, CT3 and CT4.[1]

[1] Observe that $\mathrm{CT}^-[\mathrm{PA}]$ does not contain any induction axioms for formulae with the truth predicate.

The theory $\mathrm{CT}^-[\mathrm{PA}] + (\mathrm{GR}(\mathrm{PA}))$ is the main subject of my talk. Depending on the time constraints we shall more or less extensively discuss the recent progress in discovering various natural principles, which are, over $\mathrm{CT}^-[\mathrm{PA}]$, equivalent to $(\mathrm{GR}(\mathrm{PA}))$.

It is known from some time that over $\mathrm{CT}^-[\mathrm{PA}]$, $(\mathrm{GR}(\mathrm{PA}))$ admits various natural reformulations.[2] We shall focus on two such principles: Σ_1-reflection over the uniform Tarski biconditionals (Σ_1-REF(UTB^-)) and one-sided disjunctive correctness. The former consists of the infinite collection of sentences

$$\forall x\big(\mathrm{Prov}_{\mathrm{PA}+\mathrm{UTB}^-}(\phi(\dot{x})) \rightarrow \phi(x)\big),$$

where $\phi(x)$ is a Σ_1 formula of the language with a truth predicate and UTB^- consists of infinitely many sentences of the form

$$\forall x\big(T(\phi(\dot{x})) \equiv \phi(x)\big),$$

where $\phi(x)$ is a formula of $\mathcal{L}_{\mathrm{PA}}$. It is a very easy observation that (over $\mathrm{CT}^-[\mathrm{PA}]$) Σ_1-REF(UTB^-) implies $(\mathrm{GR}(\mathrm{PA}))$. The reverse implication was established in [5], thus answering an open question posed in [1]. Σ_1-REF(UTB^-) is conceptually the strongest principle known thus far to be equivalent to $(\mathrm{GR}(\mathrm{PA}))$.

The latter principle, one-sided disjunctive correctness, called also DC-out, is the sentence

$$\forall x\big(\mathrm{SetSent}(x) \rightarrow T(\bigvee x) \rightarrow \exists \phi \in x T(\phi)\big),$$

where $\mathrm{SetSent}(x)$ expresses that x is a set of sentences (of arbitrary finite cardinality) and $\bigvee x$ denotes the disjunction (parenthesized in a canonical way) of elements in x. Thus, DC-out is a (one side of a) natural generalization of CT2 to disjunctions of arbitrary length and expresses that each such disjunction, if true, must have a true disjunct. The equivalence between DC-out and $(\mathrm{GR}(\mathrm{PA}))$ was demonstrated in [2] and refined an earlier result from [3]. Conceptually, DC-out is "the weakest" statement known to be equivalent to $(\mathrm{GR}(\mathrm{PA}))$.

References

[1] Lev D. Beklemishev and Fedor N. Pakhomov. Reflection algebras and conservation results for theories of iterated truth. *https://arxiv.org/abs/1908.10302*, 2019.

[2] Cezary Cieśliński, Mateusz Łełyk, and Bartosz Wcisło. The two halves of disjunctive correctness. *unpublished draft*.

[3] Ali Enayat and Fedor Pakhomov. Truth, disjunction, and induction. *Arch. Math. Logic*, 58(5-6):753–766, 2019.

[4] Mateusz Łełyk. *Axiomatic Theories of Truth, Bounded Induction and Reflection Principles*. PhD thesis, 2017. depotuw.ceon.pl/handle/item/2266.

[5] Mateusz Łełyk. Model theory and proof theory of the global reflection principle. *unpublished draft, https://sites.google.com/uw.edu.pl/lelyk/research*, 2021.

[2] In particular, as shown in [4] and [5], it is equivalent to Δ_0-induction for the extended language.