Indiscernibles and Satisfaction Classes in Arithmetic

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Indiscernibles

- Indiscernibles were introduced in model theory in the celebrated work of Ehrenfeucht and Mostowski 1956. The motivation for their work was a question of Hasenjaeger: Is there a model of true arithmetic that admits a nontrivial automorphism?
- **Theorem.** (Ehrenfeucht and Mostwoski). *Every first order theory with an infinite model has a model that admits a nontrivial automorphism.*
- **Definition.** Given an $\mathcal{L}$-structure $\mathcal{M}$, and some linear order $(I,<)$ where $I \subseteq M$, we say that $(I,<)$ is a set of order indiscernibles in $\mathcal{M}$ if for any $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, and any two n-tuples $\bar{i}$ and $\bar{j}$ from $[I]^n$, we have:
$$\mathcal{M} \models \varphi(i_1, \cdots, i_n) \leftrightarrow \varphi(j_1, \cdots, j_n).$$
- **Theorem.** (Ehrenfeucht and Mostowski). *Given a first order theory T with an infinite model, and any linearly ordered set $(X,<)$, there is a model M of T that has a copy of $(X,<)$ as order indiscernibles.*
- Indiscernibles have proved to be pervasive in both model theory, and indispensible in the study of large cardinals in set theory.

Indiscernibles and PA

- Every extension of PA has a model that carries no pair of indiscernibles. Indeed such models can be arranged to be of any infinite power $\leq 2^{\aleph_0}$, using "Gaifman's machinery".
- Every recursively saturated model $\mathcal{M}$ of PA (of any cardinality) carries an infinite set of indiscernibles.
- Indiscernibles naturally arise in models of PA obtained by "iterating a Gaifman minimal type".
- By a 1982 theorem of Schmerl, which answered a question of Macintyre, given a countable recursively satuated model $\mathcal{M}$ of PA, we can even find a set of order indiscernibles that generate $\mathcal{M}$ (via the definable terms).

Axioms of PAI

Let $\mathcal{L}_{\mathrm{PA}}(I) = \mathcal{L}_{\mathrm{PA}} \cup \{I\}$, where I is a unary predicate.
PAI is the theory formulated in $\mathcal{L}_{\mathrm{PA}}(I)$ whose axioms consist of the three groups below.

- Note that we often write $x \in I$ instead of $I(x)$.

(1) PA^*, i.e., $\mathrm{PA}(\mathcal{L})$ for $\mathcal{L} = \mathcal{L}_{\mathrm{PA}}(I)$.

(2) The sentence expressing "I is a unbounded in the universe" .

(3) The scheme $\mathrm{Indis}(I) = \{\mathrm{Indis}_{\varphi}(I) : \varphi$ is a formula of $\mathcal{L}_{\mathrm{PA}}\}$. More explicitly, for each n-ary formula $\varphi(v_1, \cdots, v_n)$ in the language of PA, $\mathrm{Indis}_{\varphi}(I)$ is the sentence:

$$\forall x_1 \in I \cdots \forall x_n \in I \ \forall y_1 \in I \cdots \ \forall y_n \in I$$

$$[(x_1 < \cdots < x_n) \wedge (y_1 < \cdots < y_n) \to (\varphi(x_1, \cdots, x_n) \leftrightarrow \varphi(y_1, \cdots, y_n))].$$

Elementary considerations (1)

- $(\mathcal{M}, I) \models \mathrm{PAI}$ iff the following three conditions are satisfied:
  (1) $(\mathcal{M}, I) \models \mathrm{PA}^*$,
  (2) I is unbounded in $\mathcal{M}$, and
  (3) $(I, <)$ is a set of order indiscernibles over $\mathcal{M}$.
- Let PAI° be the weakening of PAI in which the scheme $\mathrm{Indis}_{\mathcal{L}_\mathrm{A}}(I)$ is weakened to the scheme $\mathrm{Indis}^\circ(I) = \{\mathrm{Indis}^\circ_\varphi(I) : \varphi \text{ is an } \mathcal{L}_{\mathsf{PA}}\text{-formula}\}$, where $\mathrm{Indis}^\circ_\varphi(I)$ is the following sentence:

$$\forall x_1 \in I \cdots \forall x_n \in I\ \forall y_1 \in I \cdots \forall y_n \in I$$
$$[(x_1 < \cdots < x_n) \wedge (y_1 < \cdots < y_n) \wedge (\ulcorner \varphi \urcorner < x_1 \wedge \ulcorner \varphi \urcorner < y_1)$$
$$\rightarrow (\varphi(x_1, \cdots, x_n) \leftrightarrow \varphi(y_1, \cdots, y_n))].$$

Elementary considerations (2)

Proposition. *Let* $\mathbb{N}$ *be the standard model of* PA.

1. $\mathbb{N}$ *does not have an expansion to a model of* PAI (*equivalently: Every model of* PAI *is nonstandard*).

2. $\mathbb{N}$ *has an expansion to* PAI°.

3. *If* $(\mathcal{M}, I)$ *is a nonstandard model of* PAI°, *and* c *is any nonstandard element of* $\mathcal{M}$, *then* $(\mathcal{M}, I^{>c}) \models \mathrm{PAI}$, *where* $I^{>c} = \{i \in I : i > c\}$.

The interpretability lemma

- **Interpretability Lemma.** Given any $\mathcal{M} \models \mathrm{PA}$, *and any finite set* F *of axioms of* PAI, *there is a parameter free definable subset* I *of* $\mathcal{M}$ *such that* $(\mathcal{M}, I) \models F$. *More succinctly: Each finite subtheory of* PAI *has an "*ω*-interpretation" in* PA.
- **Corollary 1.** PAI *is a conservative extension of* PA.
- **Corollary 2.** PAI *is interpretable in* PA, *hence* PA *and* PAI *are mutually interpretable. But they are not bi-interpretable.*
- **Corollary 3.** PAI *is interpretable in* ACA_0, *but not vice versa.*

Satisfaction classes and Truth classes

- Let $\mathrm{Sat}(S, x)$ be a formula in the language $\mathcal{L}_{\mathrm{PA}} \cup \{S\}$ (where S is a binary predicate) that expresses "S satisfies Tarski's compositional clauses for all formulae of length $\leq x$".
- UTB is the theory formulated in $\mathcal{L}_{\mathrm{PA}} \cup \{T\}$ (where T is a unary predicate) whose axioms consist of PA* plus uniform Tarski biconditionals, i.e., sentences of the form $\forall x[\varphi(x) \leftrightarrow T(\ulcorner\varphi(\dot{x})\urcorner)]$, as φ ranges in the metatheory over arithmetical formulae.
- Given a nonstandard model $\mathcal{M}$ of PA, and a subset S of M, we say that S is a partial inductive satisfaction class if $(\mathcal{M}, S) \models \mathrm{PA}^*$ and for some nonstandard $c \in M$, $(\mathcal{M}, S) \models \forall i < c\ \mathrm{Sat}(S, i)$.
- **Folklore Proposition.** *A nonstandard model $\mathcal{M}$ of* PA *carries a partial inductive satisfaction class iff $\mathcal{M}$ has an expansion to* UTB.
- **Theorem** (Barwise and Schlipf 1978). *Suppose $\mathcal{M}$ is a model of* PA.
(1) *If $\mathcal{M}$ is nonstandard (of any cardinality) and expandable to* UTB, *then $\mathcal{M}$ is recursively saturated.*
(2) *If $\mathcal{M}$ is countable and recursively saturated, then $\mathcal{M}$ has an expansion to* UTB.

Theorem A

Theorem A. *A nonstandard model $\mathcal{M}$ of PA (of any cardinality) has an expansion to a model of PAI iff $\mathcal{M}$ carries a partial inductive satisfaction class.*

Proof. We first verify the right-to-left direction. Suppose S is a partial inductive satisfaction class over $\mathcal{M}$. Consider the formula $\varphi(S, x)$ in the extended language, where the predicate S is added to $\mathcal{L}_{\mathrm{PA}}$, that expresses:

"there is a definable (in the sense of S) unbounded homogeneous set for all $\mathcal{L}_{\mathrm{PA}}$-formulae of length at most x".

By the schematic provability of Ramsey's theorem in PA, for each $n \in \omega$, $(\mathcal{M}, S) \models \varphi(n)$, so by overspill, $(\mathcal{M}, S) \models \varphi(c)$ holds for some nonstandard $c \in M$. Hence there is an unbounded subset I of M that is indiscernibles over $\mathcal{M}$ such that I is parametrically definable in $(\mathcal{M}, S)$, thus $(\mathcal{M}, I) \models \mathrm{PAI}$.

The above argument first appeared in a 1982 paper of Roman Kossak.

Tools needed for for the other direction of Theorem A (1)

- For each $n+1$-*ary* arithmetical formula $\varphi(\overline{x}, y)$, Apart_φ is the following $\mathcal{L}_{\mathrm{PA}}(I)$ formula:

$$\forall i \in I \; \forall j \in I \, [i < j \to \forall x_1, \cdots, x_n < i \;\; (\exists y \varphi(\overline{x}, y) \to \exists y < j \; \varphi(\overline{x}, y))].$$

- **Apartness Lemma**. *For every arithmetical formula* φ,

$$\mathrm{PAI} \vdash \mathrm{Apart}_\varphi.$$

- Thus in a model of PAI, IF $i < j$ are both in I and $f(\overline{x})$ is an arithmetically definable function, THEN $f(\overline{a}) < j$ for every $\overline{a} < i$.

Tools needed for the other direction of Theorem A (2)

- Suppose $\varphi(\overline{x}, z_0, z_1, \cdots, z_r)$ be an $(n+1+r)$-ary arithmetical formula.
- Let $\mathrm{Indis}_{\varphi}^{+}$ be the following sentence of $\mathcal{L}_{\mathrm{PA}}(I)$:

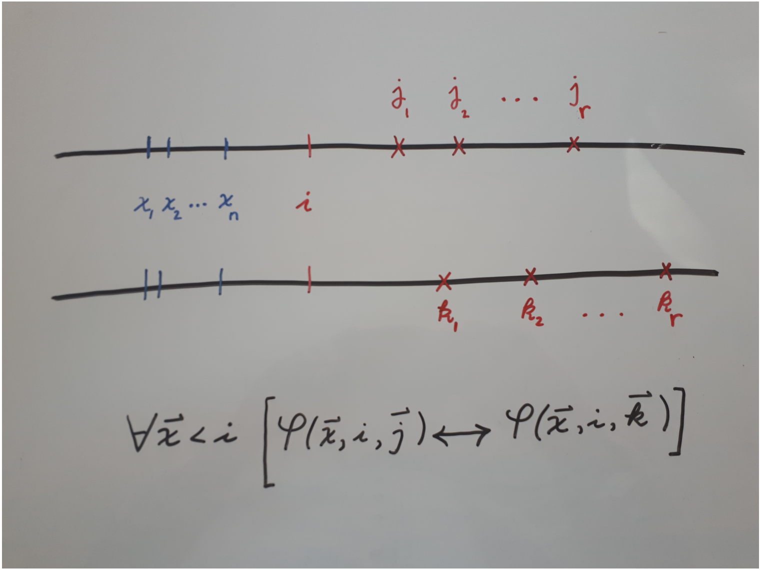
$$\forall i \in I \ \forall \overline{j} \in [I]^r \ \forall \overline{k} \in [I]^r \ \ [(i < j_1) \wedge (i < k_1)] \longrightarrow$$

$$[\forall x_1, \cdots, x_n < i \ (\varphi(\overline{x}, i, j_1, \cdots, j_r) \leftrightarrow \varphi(\overline{x}, i, k_1, \cdots, k_r))] \, .$$

- **Diagonal Indiscernibility Lemma**. *For every arithmetical formula* φ,

$$\mathrm{PAI} \vdash \mathrm{Indis}_{\varphi}^{+}.$$

Picture for diagonal indiscernibility



Tools needed for the other direction of Theorem A (3)

Theorem. *There is a formula* $\sigma(x)$ *in the language* $\mathcal{L}_{\mathrm{PA}}(I)$ *such that for all models* $(\mathcal{M}, I) \models \mathrm{PAI}$, $\sigma^{\mathcal{M}}$ *is an inductive partial satisfaction class on* $\mathcal{M}$.

Proof. We first define a recursive function that transforms each formula $\varphi(\overline{x}) \in \mathrm{Form}_n(\mathcal{L}_{\mathrm{PA}})$ into a Δ_0-formula $\varphi^*(\overline{x}, z_1, \cdots, z_k)$, where $\{z_n : 1 \leq n \in \omega\}$ is a fresh supply of variables added to the syntax of first order logic (the definition of φ^* below will make it clear that k is the $\exists$-depth of φ). In what follows x and y range over the set of variables before the addition of the fresh stock of z_ns. We assume that the only logical constants used in φ are $\{\neg, \vee, \exists\}$ and none of the fresh variables z_n occurs in φ.

(1) If φ is atomic, then $\varphi^* = \varphi$.

(2) $(\neg\varphi)^* = \neg\varphi^*$.

(3) $(\varphi_1 \vee \varphi_2)^* = \varphi_1^* \vee \varphi_2^*$.

(4) $(\exists y\ \varphi)^* = \exists y < z_1\ \widetilde{\varphi^*}$, where $\varphi^* = \varphi^*(\overline{x}, y, z_1, \cdots, z_k)$, and $\widetilde{\varphi^*}$ is the result of replacing z_i with z_{i+1} in φ^* for each $1 \leq i \leq k$.

Another view of the transformation $\varphi \mapsto \varphi^*$

- The transformation $\varphi \mapsto \varphi^*$ can be reformulated as follows: Given $\varphi(\overline{x}) \in \mathrm{Form}_n(\mathcal{L}_{\mathrm{PA}})$, first find an equivalent formula $\varphi'(\overline{x})$ in the prenex normal form:

$$\varphi'(\overline{x}) = \forall v_1 \exists w_1 \cdots \ \delta(v_1, w_1 \cdot \cdot, v_k, w_k, \overline{x}),$$

where $\delta \in \Delta_0$, and then define $(\varphi(\overline{x}))^*$ to be:

$$\forall v_1 < z_1 \exists w_1 < z_2 \cdots \ \delta(v_1, w_1, \cdots, v_k, w_k, \overline{x}).$$

- A similar transformation is found in the proof of the Paris-Harrington Theorem.

Tools needed for the other direction of Theorem A (4)

Lemma. *Suppose* $\varphi = \varphi(\overline{x}) \in \mathrm{Form}_n(\mathcal{L}_{\mathrm{PA}})$, and $\varphi^* = \varphi^*(\overline{x}, z_1, \cdots, z_k)$, $(\mathcal{M}, I) \models \mathrm{PAI}$, $\overline{a} \in M^n$, *and* $(i_1, \cdots, i_k) \in [I]^k$ *such that there is some* $j \in I$ *with* $j < i_1$ and each $a_i < j$ *Then*:

$$\mathcal{M} \models \varphi(\overline{a}) \longleftrightarrow \varphi^*(\overline{a}, i_1, \cdots, i_k).$$

The following definition takes place in $(\mathcal{M}, I)$: Given any $\varphi(\overline{x}) \in \mathrm{Form}_n(\mathcal{L}_{\mathrm{PA}})$ and any n-tuple $\overline{a}$, calculate $(\varphi(\overline{x}))^* = \varphi^*(\overline{x}, z_1, \cdots, z_k)$, and let $j \in I$ be the first element of I such that each $a_i < j$, and then let and $i_1, \cdots, i_k$ to be the first k elements of I that are above j. Then define S by:

$$\varphi(\overline{a}) \in S \text{ iff } \varphi^*(\overline{a}, i_1, \cdots, i_k) \in \mathrm{Sat}_{\Delta_0},$$

where Sat_{Δ_0} is the canonical Σ_1-definable satisfaction predicate for Δ_0 formulae of arithmetic.

S is an inductive partial satisfaction class by the lemma. QED (Theorem A).

Corollaries of Theorem A

Corollary. *Suppose* $\mathcal{M} \models \mathrm{PA}$.

(a) *There is no parametrically* $\mathcal{M}$*-definable subset* I *of* M *such that* $(\mathcal{M}, I) \models \mathrm{PAI}$. *Therefore no rather classless model of* PA *has an expansion to a model of* PAI.

(b) *If* $\mathcal{M}$ *has an expansion to a model of* PAI, *then* $\mathcal{M}$ *is recursively saturated; and the converse holds if* $\mathcal{M}$ *is countable.*

(c) *If* $\mathcal{M}$ *has an expansion* $(\mathcal{M}, I) \models \mathrm{PAI}$, *then* $M \neq M_I$, *where* M_I *consists of elements of* M *that are definable in* $(\mathcal{M}, i)_{i \in I}$, in constrast with Schmerl's result from the first page.

Remark. *Every countable recursively saturated model* $\mathcal{M}$ *of* PA *has an expansion* $(\mathcal{M}, I) \models \mathrm{PAI}$ *such that* $(\mathcal{M}, I)$ *is pointwise definable.*

Preparations for Theorem B

- S is an inductive full satisfaction class on a model $\mathcal{M}$ of PA if $(\mathcal{M}, S) \models \mathrm{PA}^*$, and S satisfies Tarski's compositional clauses for a truth predicate for all arithmetical formulae in the sense of $\mathcal{M}$.This corresponds to the truth predicate in the truth theory known as CT (compositional truth with full induction).
- Given a recursively axiomatized theory T extending $\mathrm{I}\Delta_0 + \mathrm{Exp}$, the *uniform reflection scheme over* T, denoted $\mathrm{RFN}(T)$, is defined via:

$$\mathrm{RFN}(T) := \{\forall x(\mathrm{Prov}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x)) : \varphi(x) \in \mathrm{Form}_1\}.$$

The sequence of schemes $\mathrm{RFN}^{\alpha}(T)$, where α is recursive ordinal α, is defined as follows:

1. $\mathrm{RFN}^0(T) = T$;
2. $\mathrm{RFN}^{\alpha+1}(T) = \mathrm{RFN}(\mathrm{RFN}^{\alpha+1}(T))$;
3. $\mathrm{RFN}^{\gamma}(T) = \bigcup_{\alpha<\gamma} \mathrm{RFN}^{\alpha}(T)$.

- **Theorem.** (Folklore) *The arithmetical consequences of* CT *are axiomatized by* $\mathrm{PA} + \mathrm{RFN}^{\varepsilon_0}(\mathrm{PA})$.

Theorem B.

- **Theorem B**. *There is a sentence α in the language obtained by adding a unary predicate $I(x)$ to the language of arithmetic such that given any nonstandard model $\mathcal{M}$ of PA of any cardinality,*

  $\mathcal{M}$ has an expansion to $\mathrm{PAI} + \alpha$ iff $\mathcal{M}$ has a inductive full satisfaction class.

Fragments of PAI

- For $n \in \omega$, PAI_n is the subsystem of PAI in which the extended induction scheme involving I is limited to $\Sigma_n(I)$-formulae, i.e., the axioms of PAI_n consist of PA plus the fragment $\mathrm{I}\Sigma_n(I)$ of $\mathrm{PA}(I)$, plus axioms asserting the unboundedness and indiscernibility of I.
- PAI^- is the subsystem of PAI_0 with no extended induction scheme involving I, so the axioms of PAI^- consist of PA plus axioms asserting the unboundedness and indiscernibility of I.
- Given $\mathcal{M} \models \mathrm{PA}$, it is evident that:
  1. $(\mathcal{M}, I) \models \mathrm{PAI}^-$ iff I is an unbounded set of indiscernibles in $\mathcal{M}$, and
  2. $(\mathcal{M}, I) \models \mathrm{PAI}_0$ iff PAI^- holds and I is piecewise-coded in $\mathcal{M}$.

Two results about fragments of PAI

- **Theorem 1.** *Every model of* PA *has an elementary end extension that has an expansion to a model of* PAI_0, *but not to a model of* PAI.

- **Theorem 2.** *If* $\mathcal{M}$ *is a model of countable cofinality of* PA *that is expandable to a model of* PAI^-, *then* $\mathcal{M}$ *is expandable to a model of* PAI_0. *However, every countable model of* PA *has an uncountable elementary end extension that is expandable to a model of* PAI^-, *but not to* PAI_0.

Questions

- **Question 1.** Does Theorem 1 lend itself to a hierarchical generalization? In other words, is it true that for every $n \in \omega$, every model of PA has an elementary end extension that has an expansion to a model of PAI_n, but not to a model of PAI_{n+1}? It is not even clear how to build a model of PAI_n for $n \in \omega$ that is not a model of PAI_{n+1}.
- **Question 2.** Is there a model $\mathcal{M}$ of PA such that $\mathcal{M}$ has an expansion to a model of PAI_n for each $n \in \omega$, BUT $\mathcal{M}$ has no expansion to a model of PAI?
- **Question 3.** Is there a set of sentences Σ in the language obtained by adding a unary predicate $I(x)$ to the language of arithmetic such that given any nonstandard model $\mathcal{M}$ of PA of any cardinality, $\mathcal{M}$ an expansion to a model of $\mathrm{PAI}^- + \Sigma$ iff $\mathcal{M}$ has a full satisfaction class?
- This talk was based on my paper with the same title on arXiv 2022.

Thank you for your attention