(Short) bounded recursions and Δ_0-definability

Henri-Alex Esbelin

Clermont Auvergne Université

JAF 42, SAMOS 2023

A major open problem

$$\Delta_0^{\mathbb{N}} \subseteq \mathcal{E}_\star^0$$

Equality or not ?

A major open problem in other terms

Q 1. Let us suppose that

$$\begin{cases} f(\vec{u},0) = u_0 \\ f(\vec{u},i+1) = h(\vec{u},i,f(i)) \end{cases}$$

and

$$\begin{cases} f(\vec{u},y) \leq Max\{\vec{u},y\} \\ \text{The graph of } h \text{ is } \Delta_0 - \text{definable} \end{cases}$$

Is the graph of f Δ_0-definable ?

A major open problem in other terms

Q 2. Find (if any) a function h with a Δ_0-definable graph such that, for

$$\begin{cases} f(\vec{u},0) = u_0 \\ f(\vec{u},i+1) = h(\vec{u},i,f(i)) \end{cases}$$

with

$$f(\vec{u},y) \leq Max\{\vec{u},y\}$$

the graph of f is not Δ_0-definable.

A major open problem in other terms

Q 3. Find a significant class of functions h with a Δ_0-definable graph such that f defined by

$$\begin{cases} f(\vec{u},0) = u_0 \\ f(\vec{u},i+1) = h(\vec{u},i,f(i)) \end{cases}$$

has a Δ_0-definable graph.

This is the question considered her.

Plan

- Basic informations on Δ_0-definability
- Bounded recursions : known results
- Main results and ideas of proofs
- Conclusion
- References

Basic informations on Δ_0–definability

- $z = x^y$ IS Δ_0–definable
- The graph of the following function f

$$\begin{cases} f(0) = 0 & & \\ f(i+1) = & (f(i)+1) \mod 2 & \text{if } i \text{ is prime} \\ f(i+1) = & f(i) & \text{if } i \text{ is not prime} \end{cases}$$

IS NOT KNOWN TO BE Δ_0–definable

- BUT the graph of $f(lh_2(x))$ IS Δ_0–definable

N.B. $lh_2(x)$ is the length of the binary representation of x.

Basic informations on Δ_0-definability

The method of proof :

Basic informations on Δ_0−definability

The method of proof :

$$\begin{cases} f(0) = 0 \\ f(i+1) = \quad (f(i)+1) \mod 2 & \text{if i is prime} \\ f(i+1) = \quad f(i) & \text{if i is not prime} \end{cases}$$

$z = f(y)$ iff

Basic informations on Δ_0−definability

The method of proof :

$$\begin{cases} f(0) = 0 \\ f(i+1) = (f(i)+1) \mod 2 & \text{if i is prime} \\ f(i+1) = f(i) & \text{if i is not prime} \end{cases}$$

$z = f(y)$ iff $(z_0, z_1, ..., z_y) \in \{0,1\}^{y+1}$ exists such that

$$\begin{cases} z_0 = 0 \\ \forall i \leq y-1 \quad \begin{cases} z_{i+1} = (z_i + 1) \mod 2 & \text{if } i \text{ is prime} \\ z_{i+1} = z_i & \text{if } i \text{ is not} \end{cases} \\ z = z_y \end{cases}$$

Basic informations on Δ_0−definability

The method of proof :

$$\begin{cases} f(0) = 0 \\ f(i+1) = (f(i)+1) \mod 2 & \text{if i is prime} \\ f(i+1) = f(i) & \text{if i is not prime} \end{cases}$$

$z = f(y)$ iff $(z_0, z_1, ..., z_y) \in \{0,1\}^{y+1}$ exists such that $\exists Z \leq 2^{y+1}$

$$\begin{cases} Z_0 = 0 \\ \forall i \leq y-1 \quad \begin{cases} Z_{i+1} = (Z_i + 1) \mod 2 & \text{if } i \text{ is prime} \\ Z_{i+1} = Z_i & \text{if } i \text{ is not} \end{cases} \\ z = Z_y \end{cases}$$

N. B. Z_i is the i-th digit of the binary representation of Z.

Basic informations on Δ_0−definability

The method of proof :

$$\begin{cases} f(0) = 0 & \\ f(i+1) = (f(i)+1) \mod 2 & \text{if i is prime} \\ f(i+1) = f(i) & \text{if i is not prime} \end{cases}$$

$z = f(y)$ iff $\exists Z \leq 2^{y+1}$

$$\begin{cases} Z_0 = 0 \\ \forall i \leq y-1 \quad \begin{cases} Z_{i+1} = (Z_i + 1) \mod 2 & \text{if } i \text{ is prime} \\ Z_{i+1} = Z_i & \text{if } i \text{ is not} \end{cases} \\ z = Z_y \end{cases}$$

N. B. If y is (bounded by) a logarithm of some variable, the quantification is bounded by a polynomial (of this variable)

Short recursions, long recursions

$$\begin{cases} \bar{f}(\vec{u}, 0) = u_0 \\ \bar{f}(\vec{u}, i+1) = h(\vec{u}, i, \bar{f}(\vec{u}, i)) \end{cases}$$

long recursions	short recursions
$\bar{f}(\vec{u}, y)$	$f(\vec{u}, x) = \bar{f}(\vec{u}, lh_2(x))$

Basic informations on Δ_0-definability and recursions

transition function		long rec	short rec
$z+1$ if $R(\vec{u},i)$, else z	R is Δ_0	$\Delta_0^\sharp$	Δ_0 [1]
az	a is a variable	Δ_0 [2]	
$z+b(\vec{u},i)$	$b(\vec{u},i) \leq polyn(\vec{u})$ $Graph(b)$ is Δ_0	$\Delta_0^\sharp$ [3]	Δ_0 [4]
$a(\vec{u},i) \times z$	$Graph(a)$ is Δ_0	Δ_0 [5]	
$(a \times z) \bmod m$	a, m are variables	Δ_0 [6]	

Basic informations on Δ_0-definability and recursions

Sequences issued from Euclid's algorithm

$$f(a,b,0) = a \qquad\qquad f(a,b,1) = b$$

$$f(a,b,i+2) = f(a,b,i) \bmod f(a,b,i+1)$$

It is essentially a short recursion and the graph of f is Δ_0-definable [7]

Basic informations on Δ_0-definability and recursions

Linear recurrence sequences

$$L(\vec{x}, i+k) = \sum_{j=0}^{k-1} a_j \times L(\vec{x}, i+j)$$

(k is a constant).

The graph of L is Δ_0-definable [8]

Main results and ideas of proofs

Result I

The short recursion with transition function

$$H(a, c, b, d, x, z, i) = \begin{cases} az + b \text{ if } x_i = 1 \\ cz + d \text{ if } x_i = 0 \end{cases}$$

defines a function with a Δ_0- definable graph.

N. B. x_i the i-th binary digit of x

Result I : some ideas of the proof

N. B. $\bar{x} \in \{0,1\}^\star$ *is the binary expansion of* x

$$\bar{x} = 0^{\alpha(x,0)} 1^{\beta(x,0)} 0^{\alpha(x,1)} \ldots 1^{\beta(x,lh_2(x)-2)} 0^{\alpha(x,lh_2(x)-1)} 1^{\beta(x,lh_2(x)-1)}$$

$$L(x,i) = \sum_{j=0}^{j=i-1} \alpha(x,j) + \beta(x,j)$$

$$L_0(x,i) = \sum_{j=0}^{j=i-1} \alpha(x,j) \qquad L_1(x,i) = \sum_{j=0}^{j=i-1} \beta(x,j)$$

Main results and ideas of proofs

$$\bar{F}(a,c,b,d,x,L(i)) =$$

$$a^{L_1(x,i)} c^{L_0(x,i)} \bar{F}(a,c,b,d,x,0)$$

$$+\frac{d}{c-1}\left(\sum_{j=0}^{j=i} a^{L_1(x,i)-L_1(x,j)} c^{L_0(x,i)-L_0(x,j+1)} (c^{\alpha(x,j)}-1)\right)$$

$$+\frac{b}{a-1}\left(\sum_{j=0}^{j=i} a^{L_1(x,i)-L_1(x,j)} c^{L_0(x,i)-L_0(x,j)} (a^{\beta(x,j)}-1)\right)$$

Main results and ideas of proofs

And similar formulas for

$$L(x,i) \leq y < L(x,i) + \alpha(x,i+1)$$

and

$$L(x,i) + \alpha(i+1) \leq y < L(x,i+1)$$

Main results and ideas of proofs

$z = L(x, i)$ is equivalent to

$$(\bar{x})_z = 1 \wedge (\bar{x})_{z-1} = 0 \wedge i = card\{j < i; (\bar{x})_i = 1 \wedge (\bar{x})_{i+1} = 0\}$$

with $i \leq lh_2(x)$

$$\alpha(x, i) + \beta(x, i) = L(x, i+1) - L(x, i)$$

$z = \beta(x, i)$ is equivalent to *without paying attention to borders !*

$$\exists u\, ((u = L(x, i+1) + 1) \wedge z = card\{j < u; (\bar{x})_i = 1 \wedge (\bar{x})_{i-1} = 1\})$$

Main results and ideas of proofs

$$\bar{F}(a, c, b, d, x, L(i)) =$$

$$a^{L_1(x,i)} c^{L_0(x,i)} \bar{F}(a, c, b, d, x, 0)$$

$$+ \frac{d}{c-1} \left(\sum_{j=0}^{j=i} a^{L_1(x,i) - L_1(x,j)} c^{L_0(x,i) - L_0(x,j+1)} (c^{\alpha(x,j)} - 1) \right)$$

$$+ \frac{b}{a-1} \left(\sum_{j=0}^{j=i} a^{L_1(x,i) - L_1(x,j)} c^{L_0(x,i) - L_0(x,j)} (a^{\beta(x,j)} - 1) \right)$$

Main results and ideas of proofs

The main step for studying the case where a, b, c, d are variables :

Lemma. the following relation is Δ_0-definable

$$\left(Z = \sum_{j=0}^{j=i-1} \gamma(x, j) \right) \wedge (i \leq lh_2(y))$$

where
$\star$ $\forall j \leq i\, (\gamma(x, j) \leq b(x, y))$
$\star$ $\log_2(b(x, y))$ is a polylog. of the variables
$\star$ the graph of γ and b are Δ_0-definable

Main results and ideas of proofs

$$\left(Z = \sum_{j=0}^{j=i-1} \gamma(x,j)\right) \wedge (i \leq lh_2(y))$$

is equivalent to :

$(i \leq lh_2(y)) \wedge Z \leq b(x,y) \times lh_2(y)$ and

$$\forall p \leq 2\log_2\left(b(x,y) \times lh_2(y)\right), p \text{ prime}$$

$$\left(Z \equiv \sum_{j=0}^{j=i-1} \gamma(x,j)\right) \mod p$$

Main results and ideas of proofs

now

$$\left(\sum_{j=0}^{j=i-1} \gamma(x,j) \right) \mod p$$

is equal to

$$\left(\sum_{k=0}^{k=p-1} k \times Card\{ j \leq i-1; \gamma(x,j) \equiv k \mod p \} \right) \mod p$$

Main results and ideas of proofs

Result II

The short recursion with transition function

$$h_{a_1,a_2}(m_1, m_2, z) = \left(a_2 \left(a_1 z \bmod m_1\right) \bmod m_2\right)$$

defines a function with a Δ_0- definable graph.

Main results and ideas of proofs

Result II : some ideas of the proof

N. B. u is the initial value of the recursion

$z = \bar{f}_{a_1,a_2}(m_1, m_2, u, y)$ and $0 \leq u \leq m_2 - 1$

is equivalent to

$\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ exists such that

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq u \leq m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z) \wedge$$

$$\forall i \leq y - 1 \; \mathbf{z}_{j+1} = h_{a_1,a_2}(m_1, m_2, \mathbf{z}_j)$$

Main results and ideas of proofs

$z' = h_{a_1,a_2}(m_1, m_2, z)$ and $0 \le z \le m_2 - 1$

is equivalent to

$0 \le z \le m_2 - 1$ and $k_1 \le m_2 - 1$ and $k_2 \le a_2 - 1$ exist such that

$$\begin{cases} 0 \le z' \le m_2 - 1 \\ z' + k_2 m_2 \le a_2(m_1 - 1) \\ a_1 a_2 z - z' = a_2 k_1 m_1 + k_2 m_2 \end{cases}$$

Main results and ideas of proofs

$z = \overline{f}_{a_1,a_2}(m_1, m_2, u, y)$ and $0 \leq x \leq m_2 - 1$

is equivalent to

Main results and ideas of proofs

Exist $\mathbf{k_1} \in \{0, 1, ..., m_2 - 1\}^y$ and $\mathbf{k_2} \in \{0, 1, ..., a_2 - 1\}^y$ and $\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ such that

$$(0 \le z \le m_2 - 1) \wedge (0 \le u \le m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z) \wedge$$

$$\forall i \le y$$

$$\begin{cases} \mathbf{z}_i + S_{\mathbf{k_2}}(i-2)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 \le (a_1 a_2)^i (m_1 - 1) \\ \mathbf{z}_i + S_{\mathbf{k_2}}(i-1)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x \end{cases}$$

$$\textit{where } S_{\mathbf{k}}(i) = \sum_{j=0}^{j=i} \mathbf{k}_{i-j} (a_1 a_2)^j$$

Main results and ideas of proofs

Exist $\mathbf{k_1} \in \{0, 1, ..., m_2 - 1\}^y$ and $\exists K_2 \leq x^\gamma$ and $\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ such that

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq u \leq m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z) \wedge$$

$$\forall i \leq y$$

$$\begin{cases} \mathbf{z}_i + S_{\mathbf{k_2}}(i-2)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 \leq (a_1 a_2)^i (m_1 - 1) \\ \mathbf{z}_i + S_{\mathbf{k_2}}(i-1)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x \end{cases}$$

N.B. if y is some logarithm of the variable x, γ is a constant

An easy and essential remark :

If $m_2 \geq 1 + a_2(m_1 - 1)$ then

$$h_{a_1,a_2}(m_1, m_2, z) = a_2\,(a_1 z \bmod m_1)$$

Δ_0-definability comes from Hesse theorem ([6] in the previous table), even for

$$h(a_1, a_2, m, z) = a_1\,(a_2 z \bmod m)$$

N.B. we suppose now $m_2 \leq a_2(m_1 - 1)$

Main results and ideas of proofs

Exist $\mathbf{k_1} \in \{0,1,...,m_2-1\}^y$ and $\exists K_2 \leq x_0^\gamma$ and $\mathbf{z} \in \{0,1,...,m_2-1\}^{y+1}$ such that

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq u \leq m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z) \wedge$$

$$\forall i \leq y$$

$$\begin{cases} \mathbf{z}_i + S_{\mathbf{k_2}}(i-2)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 \leq (a_1 a_2)^i (m_1 - 1) \\ \mathbf{z}_i + S_{\mathbf{k_2}}(i-1)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x \end{cases}$$

Main results and ideas of proofs

$(0 \leq z \leq m_2 - 1) \wedge$

$\forall i \leq y$

$$\left\{ \begin{array}{l} \\ \mathbf{z}_i + a_2 S_{\mathbf{k_1}}(i-1) m_1 = (a_1 a_2)^i x - S_{\mathbf{k_2}}(i-1) m_2 \end{array} \right.$$

N.B. as $\mathbf{z}_i \leq m_2 - 1 \leq a_2(m_1 - 1) - 1$

$\mathbf{z}_i$ *is a remainder and* $S_{\mathbf{k_1}}(i-1)$ *a quotient*

Main results and ideas of proofs

$$\exists K_2 \le x_0^{\gamma}$$

$$(0 \le z \le m_2 - 1) \wedge (0 \le x \le m_2 - 1)$$

$$\forall i \le y \quad \exists \zeta \le m_2 - 1 \exists \chi \le (m_2 - 1)\frac{(a_1 a_2)^{i+1} - 1}{a_1 a_2 - 1}$$

$$\begin{cases} \zeta + \chi m_2 + a_2 S_{\mathbf{k_2}}(i-1) m_1 \le (a_1 a_2)^i (m_1 - 1) \\ \zeta + \chi m_2 + a_2 S_{\mathbf{k_2}}(i-1) m_1 = (a_1 a_2)^i x \end{cases}$$

where $S_{\mathbf{k_2}}(i) = \sum_{j=0}^{j=i} \mathbf{k_2}_{i-j} (a_1 a_2)^j = \left\lfloor \frac{K_2}{(a_1 a_2)^{i+1}} \right\rfloor$

and $\zeta = ((a_1 a_2)^i x - m_2 S_{\mathbf{k_2}}(i-1)) \mod (a_2 m_1)$

and $\chi = \left\lfloor \frac{(a_1 a_2)^i x - m_2 S_{\mathbf{k_2}}(i-1)}{a_2 m_1} \right\rfloor$

Some generalizations and conclusion

Consequence 1 :

The short recursion for transition function

$$h_{R,(a,c),(b,d)}(\vec{u}, z, i) = \begin{cases} az + b \text{ if } R(\vec{u}, i, z) \\ cz + d \text{ if } \neg R(\vec{u}, i, z) \end{cases}$$

defines a function with a Δ_0-definable graph.

Consequence 1 : some ideas of the proof

The idea is that if we define a relation R' as

$$R'(\vec{u}, i) \text{ iff } R(\vec{u}, i, \bar{F}_{R,(a,c),(b,d)}(\vec{u}, i))$$

then for all $0 \leq i \leq y$, we have

$$\bar{F}_{R,(a,c),(b,d)}(\vec{u}, i) = \bar{f}_{Id,R',(a,c),(b,d)}(\vec{u}, i)$$

Some generalizations and conclusion

$z = \bar{F}_{R,i,(a,c),(b,d)}(\vec{u}, lh_2(x))$ is equivalent to

$$\exists m \in \{0,1\}^{lh_2(x)} \, (\forall i)_{i \leq lh_2(x)} \left[R'(\vec{u}, i, \bar{f}_{Id,R_m,(a,c),(b,d)}(\vec{u}, i)) \leftrightarrow m_i = 1\right]$$

$$\wedge \left[z = \bar{f}_{R_m,(a,c),(b,d)}(\vec{u}, lh_2(x))\right]$$

where $R_m(i)$ is define by $m_i = 1$.

Some generalizations and conclusion

Variant :

The long recursion for transition function

$$h_R(a, c, b, d, \vec{u}, z, i) = \begin{cases} az + b \text{ if } R(\vec{u}, i, z) \\ cz + d \text{ if } \neg R(\vec{u}, i, z) \end{cases}$$

defines a function with a $\Delta_0^\sharp$-definable graph.

Some generalizations and conclusion

Consequence :

The short recursion for transition function

$$h_{R,(a,c),(b,d)}(\vec{u}, i, z) = \begin{cases} a(\vec{u})z + b(\vec{u}) \text{ if } R(\vec{u}, i, z) \\ c((\vec{u})z + d((\vec{u}) \text{ if } \neg R(\vec{u}, i, z) \end{cases}$$

defines a function with a Δ_0-definable graph.

Generalization (work in progress)

The short recursion for transition function
$h_{(R_1,R_2,\ldots,R_k),(a_1,c_1),(a_2,c_2),\ldots,(a_k,c_k)}(\vec{u},i,z) =$

$$\begin{cases} a_1(\vec{u})z + b_1(\vec{u}) \text{ if } R_1(\vec{u},i,z) \\ \ldots. \\ a_k(\vec{u})z + b_k(\vec{u}) \text{ if } R_k(\vec{u},i,z) \end{cases}$$

defines a function with a Δ_0-definable graph.

Some generalizations and conclusion

Generalization of the second result.

The short recursion for transition function

$$h_{a_1,b_1,a_2,b_2}(m_1, m_2, z) = (a_2\,((a_1 x + b_1) \bmod m_1) + b_2) \bmod m_2$$

defines a function with a Δ_0-definable graph.

Conclusion

A question is now to give a natural characterization of a class of functions h with a Δ_0-definable graph such that f defined by

$$\begin{cases} f(\vec{u}, 0) = u_0 \\ f(\vec{u}, i+1) = h(\vec{u}, i, f(i)) \end{cases}$$

has a Δ_0-definable graph.

References

[1] and [4]) A. Woods, *Some problems in logic and number theory and their connections*, Ph.D. thesis, University of Manchester, Manchester, 1981, in *Studies in Weak Arithmetics, Vol. 2*, ed. Edited by P. Cégielski, Ch. Cornaros, and C. Dimitracopoulos, CSLI Lecture Notes (211), 2013.

[2] J. Bennett, *On spectra*, Ph.D. thesis, Princeton University, Princeton, New Jersey, USA, 1962, and H. Gaifman and C. Dimitracopoulos. *Fragments of Peano's arithmetic and the MRDP Theorem*, in Logic and Algorithmic, Geneve, pp. 187-206, 1982.

[3] H.-A. E. and M. More, *Rudimentary relations and primitive recursion : a toolbox*, T. C. S., vol. 193, n. 1-2, pp. 129-148, 1998.

References

[5] A. Berarducci & P. D'Aquino, *Δ_0-complexity of the relation $y = \prod_{i \leq n} F$*, APAL 75 (1-2) :49-56, 1995.

[6] W. Hesse, E. Allender, D.A.M. Barrington, *Uniform constant-depth threshold circuits for division and iterated multiplication*, J. Comput. Syst. Sci. 65, pp. 695-716, 2002.

[7] H.-A. E. *Δ_0-definability of the denumerant with one plus three variables*, in *Studies in Weak Arithmetics, ... Vol. 2* (opt. cit.)

[8] H.-A. E. & M. More, (opt. cit.) and independently A. Berarducci & B. Intrigila *Linear Recurrence Relations are Δ_0 Definable* in *Logic and Foundations of Mathematics Selected Contributed Papers of the Tenth International Congress of Logic, Methodology and Philosophy of Science, Florence, 1995*, 1999.