

Non-elementary cofinal extensions of models of fragments of Peano arithmetic

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Ongoing joint work with Wong Tin Lok

This talk:

- ▶ Focus on cofinal extensions for models of strong fragments of arithmetic, especially the elementarity of extensions.
- ▶ Motivating Question: What 'controls' the elementarity of a cofinal extension?
Answer: Second-order theory of a definable cut.

Plan:

- ▶ Introduction.
- ▶ Non-elementary cofinal extension for countable model.
- ▶ A systematic way to 'compress' truth.

First-order Arithmetic

- ▶ The language of First-order Arithmetic
 $\mathcal{L}_1 = \{+, \times, <, =, 0, 1\}$.
- ▶ A formula is Δ_0 if all its quantifiers are bounded
i.e., in the form of $\exists x < t(\bar{y}) \phi(x, \bar{y})$ or $\forall x < t(\bar{y}) \phi(x, \bar{y})$.
- ▶ $\Sigma_n = \{\exists \bar{x}_1 \forall \bar{x}_2 \dots Q \bar{x}_n \theta(\bar{x}_1, \dots, \bar{x}_n, \bar{a}) \mid \theta \in \Delta_0\}$,
 $\Pi_n = \{\forall \bar{x}_1 \exists \bar{x}_2 \dots Q \bar{x}_n \theta(\bar{x}_1, \dots, \bar{x}_n, \bar{a}) \mid \theta \in \Delta_0\}$.
- ▶ A formula is Δ_n if it is equivalent to both a Σ_n and a Π_n formula (over some model or theory).
- ▶ $I\Sigma_n$ consists of PA^- and the **Induction** for all Σ_n formula ϕ :
$$\phi(0, \bar{c}) \wedge (\forall x (\phi(x, \bar{c}) \rightarrow \phi(x + 1, \bar{c}))) \rightarrow \forall x \phi(x, \bar{c}).$$
- ▶ $B\Sigma_n$ consists of $I\Sigma_0$ and the **Collection** for all Σ_n formula ϕ :
$$\forall x < a \exists y \phi(x, y, \bar{c}) \rightarrow \exists b \forall x < a \exists y < b \phi(x, y, \bar{c}).$$
- ▶ exp asserts the totality of exponential function.
- ▶ $\text{PA} = \bigcup_{n \in \omega} I\Sigma_n$.

First-order Arithmetic

Theorem (Paris–Kirby 1978)

$$I\Delta_0 + \text{exp} \vdash B\Sigma_1 + \text{exp} \vdash I\Sigma_1 \vdash B\Sigma_2 \vdash I\Sigma_2 \vdash B\Sigma_3 \dots$$

and *none of the converses holds*.

Definition

We call models of $B\Sigma_n + \neg I\Sigma_n$ **B-Models**, and models of $I\Sigma_n + \neg B\Sigma_{n+1}$ **I-Models**.

Cofinal Extensions vs Elementarity

Definition

Let $M, K \models \text{PA}^-$, $M \subseteq K$

- ▶ K is a **cofinal extension** of M ($M \subseteq_{\text{cf}} K$) if

$$\forall x \in K \quad \exists y \in M \quad K \models x < y.$$

The extension is proper if $K \neq M$.

- ▶ For each $n \in \omega$, K is a **n -elementary extension** of M ($M \preceq_n K$) if for every $\phi \in \Sigma_n$, $\bar{a} \in M$,

$$M \models \phi(\bar{a}) \iff K \models \phi(\bar{a}).$$

Convention: We assume all models mentioned satisfy $\text{I}\Delta_0 + \text{exp}$. Since $\text{I}\Delta_0 + \text{exp} \vdash \text{MRDP}$, we have 0-elementarity for any extension.

Cofinal Extensions vs Elementarity

Theorem (Gaifman–Dimitracopoulos 1980)

Let $n \in \omega$, $M, K \models \text{PA}^-$, $M \subseteq_{\text{cf}} K$, if $M \models \text{B}\Sigma_{n+1}$, then $M \preceq_{n+2} K$. In particular, if $M \models \text{PA}$, then $M \preceq K$.

Theorem (Kaye 1991)

- ▶ For each $n \in \omega$, every countable model of $\text{B}\Sigma_{n+1} + \text{exp} + \neg\text{I}\Sigma_{n+1}$ admits a proper elementary cofinal extension.
- ▶ (Informally) Every sufficiently saturated countable model admits a proper elementary cofinal extension.

Question: How much non-elementarity can we get for cofinal extensions?

Non-elementary Cofinal Extension

Theorem

For every *countable* model $M \models I\Delta_0 + \text{exp}$, if $M \not\models \text{PA}$ then M admits a non-elementary cofinal extension.

- ▶ (Paris 1981, Friedman independently.) Every countable model $M \models I\Sigma_n + \text{exp} + \neg B\Sigma_{n+1}$ admits a cofinal extension $K \models B\Sigma_{n+1}$. So we only need to consider B-models. ($M \models B\Sigma_{n+1} + \neg I\Sigma_{n+1}$.)
- ▶ We can further require that there is $M \preceq L$ such that $M \subseteq_{\text{cf}} K \subseteq_e L$ and $M \not\preceq K$. This answers a question negatively in Kaye's paper 'On Cofinal Extensions of Models of Fragments of Arithmetic' (1991).

Corollary

A countable model of $I\Delta_0 + \text{exp}$ satisfy PA if and only if all of its cofinal extensions are fully elementary.

What Does $\neg I\Sigma_{n+1}$ Provide Us?

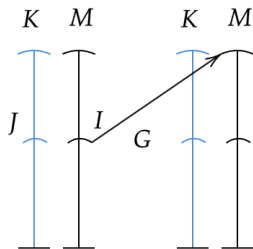
Fact

If $M \models \neg I\Sigma_{n+1}$, then:

- ▶ There is a Σ_{n+1} -definable proper cut $I \subseteq_e M$.
- ▶ There is a $\Sigma_n \wedge \Pi_n$ -definable non-decreasing total function $G: I \rightarrow M$, whose range is cofinal in M .

Suppose $M \models B\Sigma_{n+1}$. Let $\phi(x, y) \in \Sigma_n \wedge \Pi_n$ defines the graph of G , then for any $K \supseteq_{cf} M$, $\exists y \phi(x, y)$ defines a cut $J \supseteq_{cf} I$.

In particular, if ω is Σ_{n+1} -definable in M then it is still Σ_{n+1} -definable in any cofinal extension.



Non-elementary Cofinal Extension for B-Model

Fix $n \in \omega$, $M \models \text{B}\Sigma_{n+1} + \text{exp} + \neg\text{I}\Sigma_{n+1}$.

$\text{SSy}_I(M) = \{A \subseteq I \mid A \text{ coded in } M\}$, $\text{SSy}(M) = \text{SSy}_\omega(M)$.

Observation

If ω is Σ_{n+1} -definable in M , then we may describe the second-order property of $(\omega, \text{SSy}(M))$ using first-order formula in M .

e.g: $0' \in \text{SSy}(M)$ can be described as

$$\exists a \forall i \in \omega (i \in a \leftrightarrow \exists s \in \omega (M_i(i) \text{ halts with computation } s)).$$

Here $M_i(i)$ means the i -th Turing Machine with input i .

Cofinally extend M to K such that $\text{SSy}(M)$ and $\text{SSy}(K)$ satisfy different second-order properties.

Non-elementary Cofinal Extension for B-Model

Lemma

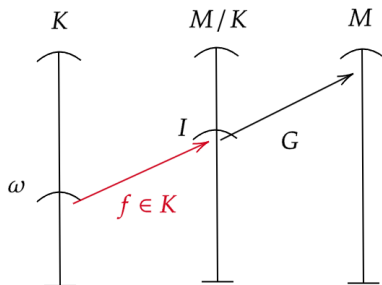
Every countable $M \models \neg I\Sigma_{n+1}$ admits a cofinal extension in which ω is Σ_{n+1} -definable.

Proof Sketch: If ω is not Σ_{n+1} -definable in M , add a coded non-decreasing function into $K \supseteq_{\text{cf}} M$:

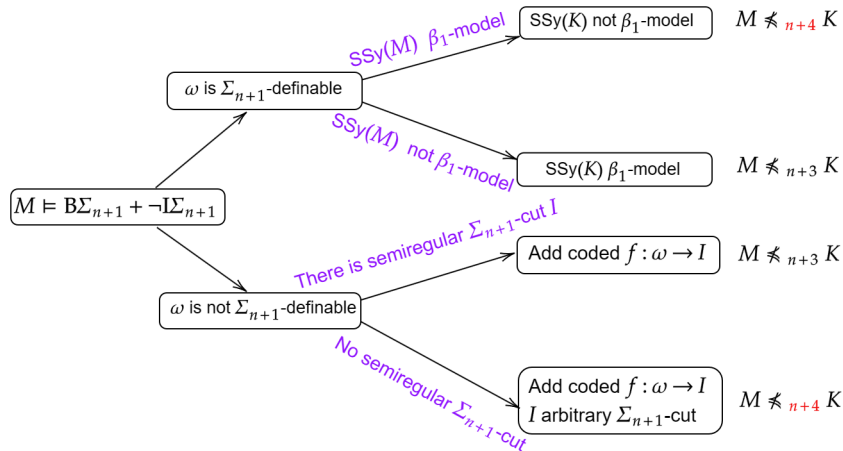
$$f: \omega \rightarrow I$$

where I is a Σ_{n+1} -definable cut in M , and $\text{Range}(f) \subseteq_{\text{cf}} I$.

Now ω is Σ_{n+1} -definable in K , and I becomes non-semiregular.



Non-elementary Cofinal Extension for B-Model



Here $SSy(M)$ is a β_1 -model means $(\omega, SSy(M)) \preceq_{\Sigma_1^1} (\omega, \mathcal{P}(\omega))$.

Question: Are the non-elementarities here optimal?

Non-elementary Cofinal Extension for B-Model

Theorem

For each $n \in \omega$:

- ▶ There is a **countable** $M \models \text{B}\Sigma_{n+1} + \text{exp} + \neg\text{I}\Sigma_{n+1}$, such that for any cofinal extension $K \supseteq_{\text{cf}} M$, $M \preceq_{n+3} K$.
- ▶ There is a **uncountable** $M \models \text{B}\Sigma_{n+1} + \text{exp} + \neg\text{I}\Sigma_{n+1}$, such that for any cofinal extension $K \supseteq_{\text{cf}} M$, $M \preceq K$.

Compressing the Truth

We just mentioned that we can describe the second-order property of $(I, \text{SSy}_I(M))$ in M if I is definable.

Actually, the converse is also true: For fixed parameter, we can describe the first-order truth of M in $(I, \text{SSy}_I(M))$ if I is Σ_{n+1} -definable.

Two ingredients for compressing

- ▶ $\neg I\Sigma_{n+1}$: Cofinal function $G: I \rightarrow M$.
- ▶ $B\Sigma_{n+1}$: Encoding Δ_{n+1} formulas.

Encoding Δ_{n+1} over a Cut

Definition (Δ_{n+1} over a Cut)

We say that $\alpha(\bar{x}, \bar{i})$ is Δ_{n+1} over $\bar{i} \in I$, if $\alpha \in \Sigma_{n+1}$ and there is some $\beta \in \Pi_{n+1}$ such that

$$M \models \forall x \forall \bar{i} \in I (\alpha(\bar{x}, \bar{i}) \leftrightarrow \beta(\bar{x}, \bar{i})).$$

Theorem (Chong–Mourad Coding Lemma 1990)

If $M \models \text{B}\Sigma_{n+1} + \text{exp}$ and I is a Σ_{n+1} -definable cut of M , $\alpha(\bar{x}, \bar{i})$ is Δ_{n+1} over $\bar{i} \in I$, then for all $b \in M$

$$M \models \exists c (\forall x < b \forall \bar{i} \in I (\langle \bar{x}, \bar{i} \rangle \in c \leftrightarrow \alpha(\bar{x}, \bar{i}))).$$

Encoding Δ_{n+1} over a Cut

We also have $B\Sigma_{n+1}$ over a Σ_{n+1} definable cut.

Theorem (Belanger–Chong–Li–Wong–Yang)

If $M \models B\Sigma_{n+1} + \text{exp}$ and I is a Σ_{n+1} -definable cut of M , $\alpha(x, i)$ is Δ_{n+1} over $i \in I$, then for all $a \in M$

$$M \models \forall x < a \exists i \in I \alpha(x, i) \rightarrow \exists b \in I \forall x < a \exists i < b \alpha(x, i).$$

Rewriting System

For simplicity, we assume that $n = 0$ and $I = \omega$. We fix some $M \models \text{BS}\Sigma_1 + \text{exp} + \neg\text{IS}\Sigma_1$, so $G: \omega \rightarrow M$ is Δ_0 -definable.

Observation 1

For any Π_2 formula $\forall a \exists b \theta(a, b)$ where $\theta \in \Delta_0$:

$$\forall a \exists b \theta(a, b)$$

$$\Leftrightarrow \forall x \in \omega \exists y \in \omega \underbrace{\forall a < G(x) \exists b < G(y) \theta(a, b)}_{\alpha(x,y) \in \Delta_1 \text{ over } x,y \in \omega} \quad \text{G is Cofinal}$$

$$\Leftrightarrow \exists f: \omega \rightarrow \omega \forall x \in \omega \alpha(x, f(x)) \quad \text{Coding Lemma}$$

$f: \omega \rightarrow \omega$ means f codes the graph of a total function from ω to ω .

Rewriting System

Temporarily let $\exists f$ abbreviates $\exists f: \omega \rightarrow \omega$.

Observation 2

For any Σ_3 formula, it is equivalent to:

$$\begin{aligned} & \exists c \forall a \exists b \theta(a, b, c) \\ \Leftrightarrow & \exists c \exists f \forall x \in \omega \alpha(x, f(x), c) && \text{Observation 1} \\ \Leftrightarrow & \exists z \in \omega \exists f \exists c < G(z) \forall x \in \omega \alpha(x, f(x), c) && \text{G is Cofinal} \\ \Leftrightarrow & \exists z \in \omega \exists f \forall x' \in \omega \underbrace{\exists c < G(z) \forall x < x' \alpha(x, f(x), c)}_{\beta(z, x', f \upharpoonright_{x'}) \in \Delta_1 \text{ over } z, x', f \upharpoonright_{x'} \in \omega} && \text{B}\Sigma_1 \text{ over } \omega \end{aligned}$$

Here fixing $z \in \omega$, those $x, f \upharpoonright_x \in \omega$ satisfy $\beta(z, x, f \upharpoonright_x)$ provides us an ω -branching tree.

Normal Form

With similar rewriting procedure(although more tedious!), we can show the following:

Theorem (Normal Form Theorem)

Let $m \in \omega$, $M \models \text{B}\Sigma_1 + \text{exp} + \neg\text{I}\Sigma_1$ and ω is Σ_1 -definable in M .
Then any Σ_{m+3} formula $\phi(\bar{c})$ in M is equivalent to the form:

$$\exists f_1 \forall f_2 \dots Q f_m \bar{Q} x \in \omega \alpha(\bar{f} \upharpoonright_x, x, \bar{c})$$

for some $\alpha \in \Delta_1$ over $\bar{f} \upharpoonright_x, x \in \omega$ effectively decided by ϕ , and such equivalence is provable in $\text{B}\Sigma_1 + \text{exp} + \neg\text{I}\Sigma_1$.

A Bit Second-order Arithmetic

Definition

A second-order formula is $\text{r}\Sigma_m^1$ if it has the form

$$\exists f_1 \forall f_2 \dots Q f_m \overline{Q} x S(x, \overline{f} \upharpoonright_x)$$

where $S \in \Delta_0^0$ and $\exists f$ abbreviates $\exists f: \omega \rightarrow \omega$.

- ▶ $\text{r}\Sigma_m^1$ and Σ_m^1 coincide over ACA_0 .
- ▶ For extensions of second-order structure, we write $(\omega, \mathcal{A}) \preceq_{\text{r}\Sigma_m^1} (\omega, \mathcal{B})$ if $\text{r}\Sigma_m^1$ formulas are absolute.

Correspondence Theorem

Lemma (Belanger–Wong)

Let $M \models \text{B}\Sigma_1 + \text{exp}$, $M \subseteq_{\text{cf}} K$, then $K \models \text{B}\Sigma_1 + \text{exp}$.

So the same normal form applied in M and K .

Theorem (First-order Second-order Correspondence)

For $M \models \text{B}\Sigma_1 + \text{exp} + \neg\text{I}\Sigma_1$, $M \subseteq_{\text{cf}} K$, ω is Σ_1 -definable in M , let $m \in \omega$,

$$M \preceq_{m+3} K \iff (\omega, \text{SSy}(M)) \preceq_{\text{r}\Sigma_{m+1}^1} (\omega, \text{SSy}(K)).$$

The correspondence helps us to convert a problem of first-order arithmetic into a problem of second-order arithmetic.

β_m -model

Definition (β_m -model)

For $m > 0$, a second-order structure (ω, \mathcal{A}) is called a β_m -model if

$$(\omega, \mathcal{A}) \preceq_{\Sigma_m^1} (\omega, \mathcal{P}(\omega)).$$

Theorem (Mummert–Simpson 2004)

For each $m > 0$, there is a countable β_m -model which is not a β_{m+1} -model.

Constructions of Cofinal Extension

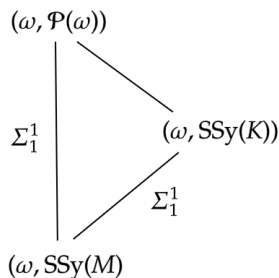
For models of $\text{B}\Sigma_1 + \text{exp} + \neg\text{I}\Sigma_1$ in which ω is Σ_1 definable:

- ▶ If $M \subseteq_{\text{cf}} K$ and $\text{SSy}(M) = \text{SSy}(K)$, then $M \preceq K$.
- ▶ For each $m \in \omega$, there is a **countable** $M \subseteq_{\text{cf}} K$ such that $M \preceq_{m+2} K$ but $M \not\preceq_{m+3} K$.
($(\omega, \text{SSy}(M))$ is a β_m -model but not a β_{m+1} -model.)

Constructions of Cofinal Extension

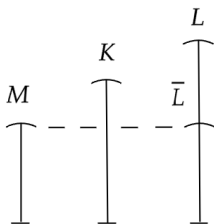
For models of $\text{B}\Sigma_1 + \text{exp} + \neg\text{I}\Sigma_1$:

- ▶ Every model M_0 admits an **uncountable** cofinal extension $M_0 \subseteq_{\text{cf}} M$, such that every cofinal extension of M is fully elementary. ($\text{SSy}(M) = \mathcal{P}(\omega)$.)
- ▶ Every **countable** M_0 admits a **countable** cofinal extension $M_0 \subseteq_{\text{cf}} M$, such that any cofinal extension of M is 3-elementary. ($(\omega, \text{SSy}(M))$ is a β_1 -model.)



Constructions of Cofinal Extension

There is a **countable** M , such that for any extension $M \subseteq K$, there is a further elementary extension $K \preceq L$ such that $M \preceq_{\text{cf}} \bar{L} \subseteq_e L$. $((\omega, \text{SSy}(M)) \preceq (\omega, \mathcal{P}(\omega).))$



This answers a main question positively in Kaye's another paper 'Model-theoretic properties characterizing Peano arithmetic' (1991).

One of the most annoying failures of this paper is the absence of a result similar to 3.7 showing (ii) to be insufficient to characterize PA over $\text{I}\Delta_0 + \text{exp}$ on its own.

PROBLEM 4.2. For all $n \geq 1$, is there a countable model $I_n \models \text{I}\Delta_0 + \text{exp} + \text{B}\Sigma_n + \neg \text{I}\Sigma_n$ such that whenever $L \succ I_n$ is countable there is $\bar{L} \succ L$ and $\bar{K} \subseteq_e \bar{L}$ such that $I_n \prec_{\text{cf}} \bar{K}$?

Answering Kaye's Questions

In the same paper, Kaye considers various model-theoretic conditions, and asks whether they characterize arithmetic theories extending PA. Here we can show that most of them fail to rule out the case of extending $\text{B}\Sigma_{n+1} + \neg\text{I}\Sigma_{n+1}$.

There are various natural variations on property (ii) of a theory T extending $\text{I}\Delta_0 + \text{exp}$ that have been considered in preliminary drafts of this paper. These include:

(iii) For each complete consistent $S \supseteq T$ there is a model $K \models S$ such that whenever $L \succ K$ there is $M \subseteq_e L$ with $K <_{\text{cf}} M$.

(iv) For each complete consistent $S \supseteq T$ there is $K \models S$ such that whenever $L \succ K$ (no restriction on $\text{card}(L)$) there is $\bar{L} \succ L$ and $\bar{K} \subseteq_e \bar{L}$ with $K <_{\text{cf}} \bar{K}$.

(v) For each consistent $S \supseteq T$ there are models K, L, M of S such that $K <_{\text{cf}} M \subseteq_e L$ with $M \neq L$ and $K < L$.

(vi) For each consistent $S \supseteq T$ there is a consistent $S^* \supseteq S$ such that whenever $K \subseteq_{\text{cf}} L$ are both models of S^* , then $K < L$.

Each of these properties is true of the theory $T = \text{PA}$; moreover, if T has any one of (iii), (iv) or (v), then $T \vdash \text{I}\Sigma_n \Rightarrow T \vdash \text{B}\Sigma_{n+1}$ for each $n \in \mathbf{N}$.

PROBLEM 4.3. If T extends $\text{I}\Sigma_n + \text{exp}$ and satisfies (vi), does $T \vdash \text{B}\Sigma_{n+1}$? Indeed, what theories $T \supseteq \text{I}\Delta_0 + \text{exp}$ (other than extensions of PA) have property (vi)?

PROBLEM 4.4. Are there theories T extending $\text{B}\Sigma_{n+1} + \text{exp} + \neg\text{I}\Sigma_{n+1}$ for some $n \in \mathbf{N}$ satisfying (iii), (iv), (v) or (vi)?

Generalized Correspondence

Theorem (First-order Second-order Correspondence, Generalized)

For $n \in \omega$, suppose $M \models \text{B}\Sigma_{n+1} + \text{exp} + \neg\text{I}\Sigma_{n+1}$ with a Σ_{n+1} -definable cut I which is closed under exponentiation in M . $M \subseteq K$ is a $(n+2)$ -elementary extension and $K \models \text{B}\Sigma_{n+1}$. Let J be the Σ_{n+1} -definable cut in K with the same definition as M , then $(I, \text{SSy}_I(M))$ naturally embed into $(J, \text{SSy}_J(K))$, and for all $m \in \omega$:

$$M \preceq_{n+m+3} K \iff (I, \text{SSy}_I(M)) \preceq_{\text{r}\Sigma_{m+n+1}^1} (J, \text{SSy}_J(K)).$$

Summary

- ▶ Any countable model of $I\Delta_0 + \exp$ fail to satisfy PA admits a non-elementary cofinal extension.
- ▶ A systematic way to ‘compress’ truth in $M \models B\Sigma_{n+1} + \neg I\Sigma_{n+1}$ in the second-order theory of its Σ_{n+1} -definable cut.
- ▶ For the case ω is Σ_{n+1} -definable, we construct models with various cofinal extension properties by considering its standard system.

Thank You!