Non-elementary cofinal extensions of models of fragments of Peano arithmetic

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A *cofinal* extension of a model of arithmetic is an extension in which every new element is bounded above by some old element. A classic result from the 1980s tells us that all such extensions are guaranteed some elementarity provided certain amount of collection.

Theorem (Gaifman–Dimitracopoulos [2]). *Let $n \in \omega$ and M be a model of the Σ_{n+1} collection scheme $\mathrm{B}\Sigma_{n+1}$. Then all Δ_0-elementary cofinal extensions of M are Π_{n+2}-elementary.*

This guaranteed level of elementarity is optimal: a simple construction gives, for each $n \in \omega$, some model of $\mathrm{B}\Sigma_{n+1}$ with a Π_{n+2}-elementary cofinal extension that is not Π_{n+3}-elementary.

A natural question that follows is: can one similarly separate higher levels of elementarity for such cofinal extensions? We answer this in the affirmative.

Theorem. *For each $m, n \in \omega$, there exists a model of $\mathrm{B}\Sigma_{n+1}$ with a Π_{m+n+2}-elementary cofinal extension that is not Π_{m+n+3}-elementary.*

To prove this theorem, we want to 'compress' the truth of a model into its initial segment. For this, we work in models of $\mathrm{B}\Sigma_{n+1} + \exp + \neg\mathrm{I}\Sigma_{n+1}$. We refine a recent quantifier elimination result by Belanger, Chong, Li, Wong and Yang [1] to rewrite arbitrary formulas into a 'second-order-like' normal form, and obtain a level-by-level correspondence between truth in a model $M \models \mathrm{B}\Sigma_{n+1}$ and second-order truth in $(I, \mathrm{SSy}_I(M))$, where I is a Σ_{n+1}-definable initial segment in M.

Theorem. *Fix $m, n \in \omega$ and $M \models \mathrm{B}\Sigma_{n+1} + \exp$. Let I be an exponentially closed Σ_{n+1}-definable proper initial segment of M, and K be a Δ_0-elementary cofinal extension of M. If J denotes the smallest initial segment of K that contains all the elements of I, then*

$$M \preccurlyeq_{\Pi_{m+n+3}} K \quad \Longleftrightarrow \quad (I, \mathrm{SSy}_I(M)) \preccurlyeq_{\mathrm{r}\Pi^1_{m+n+1}} (J, \mathrm{SSy}_J(K)),$$

where $\mathrm{r}\Pi^1_{m+n+1}$ is the set of all Π^1_{m+n+1} formulas in Kleene Normal Form.

This allows us to convert a problem of models of first-order arithmetic into a problem of models of second-order arithmetic, and we may apply powerful machineries from second-order arithmetic to construct the required cofinal extensions.

Actually the correspondence above can be generalized to arbitrary extensions under certain assumptions. There are also a number of other interesting consequences from the correspondence.

References

[1] D. Belanger, C. T. Chong, W. Li, T. L. Wong, and Y. Yang. Quantification over higher types in B-arithmetic. In preparation.

[2] H. Gaifman and C. Dimitracopoulos. Fragments of Peano's arithmetic and the MRDP theorem. In *Logic and algorithmic*, volume 30 of *Monograph. Enseign. Math.*, pages 187–206. Université de Genève, Geneva, 1982.

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