Infinite Time Turing Machines for elementary proofs on recursive reals

Journée des Arithmétiques Faibles

Kenza Benjelloun

Joint work with Bruno Durand

Université de Côte d'Azur

Università Degli Studi di Trieste

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Outline of talk

Preliminary notions

Infinite Time Turing machines and algorithmic tools

Our elementary proof of Harrison’s theorem

Prooving that our proof is elementary

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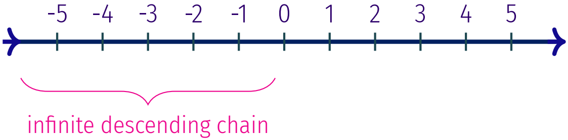
Order relations

$\mathbb{Z}$:



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We denote orders by $\prec$ here $a \prec b \iff a <_{\mathbb{Z}} b$

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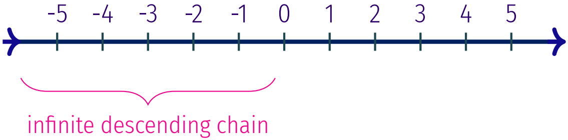

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This is an order (strict)

- *Anti-reflexive* ($x \not\prec x$)
- *Linear* (any two elements are comparable)
- *Transitive* ($\forall abc\ a \prec b \wedge b \prec c \Rightarrow a \prec c$)

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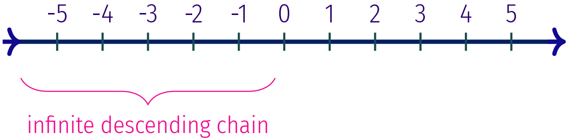

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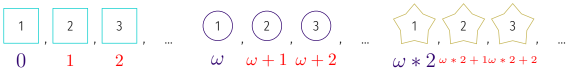
- ∘ *Linear* (any two elements are comparable)
- ∘ *Transitive* ($\forall abc\ a \prec b \wedge b \prec c \Rightarrow a \prec c$)
- × **Not** a *well-order* – $\mathbb{Z}$ has no least element

- well-orders : any non-empty subset has a least element
- well-order $\iff$ no infinite descending chain (Zorn - Choice)

Orders and real codes

- Ordering *countable* sequences of elements...



$$1-\frac{1}{2^1} < 1-\frac{1}{2^2} < 1-\frac{1}{2^3} < \dots \quad 2-\frac{1}{2^1} < 2-\frac{1}{2^2} < 2-\frac{1}{2^3} < \dots \quad 3-\frac{1}{2^1} < 3-\frac{1}{2^2} < 3-\frac{1}{2^3} < \dots$$

$$0 \quad 1 \quad 2 \quad \omega \quad \omega+1 \quad \omega+2 \quad \omega*2 \quad \omega*2+1 \quad \omega*2+2$$

Orders and real codes

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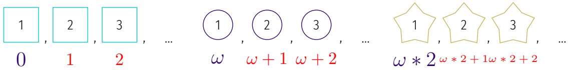

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- The objects we order are represented by integers
- Encoding pairs : $\langle a, b\rangle$ is an integer too. e.g. using *Cantor's pairing function* :

$$\langle a, b\rangle = \frac{(a+b)(a+b+1)}{2} + b$$

Orders and real codes

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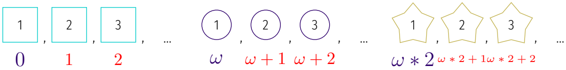

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0	1	2	ω	$\omega+1$	$\omega+2$	$\omega * 2$	$\omega * 2+1$	$\omega * 2+2$

- The objects we order are represented by integers
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$$\langle a, b\rangle = \frac{(a+b)(a+b+1)}{2} + b$$

- We encode relations by reals

$$x = 0100010101010101...0 \iff (x_{\langle a,b\rangle} = 1 \Leftrightarrow a \prec b)$$

⟶ This real encodes a relation, an order, an ordinal

Ordinals encoded as reals

Let x be a real code,
• x is **total** if all pairs of integers $n = \langle a, b \rangle$ are comparable
• the **domain** of x is the set of integers appearing in at least one comparable pair $a \prec_x b$ of the order encoded by x
◦ well-orders are always comparable – an **ordinal** is a class of well orders of same length (length is called *ordinal type*).

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Proposition

We can recursively transform any (not-total) enumerable (infinite) code into a total enumerable code

• x is a **recursive real** if $n \mapsto x_n$ is computable by a Turing machine

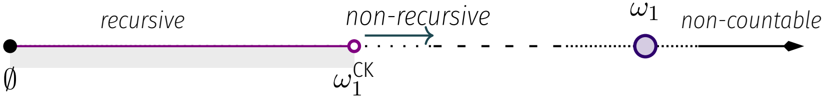
• x is an **enumerable real** if 1's in x are enumerable by a Turing machine

• α is a **recursive ordinal** if it has at least one recursive encoding

recursive ordinals

Recursive ordinals form an **initial segment** of countable ordinals :

$$\sup_{\alpha \in \mathsf{REC}} \alpha = \omega_1^{\mathsf{CK}}$$



Theorem (Spector Theorem, 1958)

The order type of a Σ_1^1 well-order is less than ω_1^{CK}

$\rightarrow$ **collapse theorem**

Collapse on recursive ordinals

Theorem (First order improved version)

If an ordinal α is arithmetic, then it is also recursive. Furthermore, the transformation of the formula that caracterizes α into a program (for $\omega \cdot \alpha$) is recursive.

$\rightarrow$ arithmetic ordinals are those ordinals encoded by a real defined by an arithmetic formula
$\rightarrow$ "α is also recursive" means that there exists another real that codes a well-order of same length α which is recursive

later we'll come back on the collapse and its role in our construction

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Infinite Time Turing Machines definition (ITTM)

- Developped by Joel David Hamkins and Andy Lewis in 2000
- Analog of a Turing machine but with ordinal time
- Tapes of size ω

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ITTM computation steps :

(1) Starting time : 0, initial state q_0

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ITTM computation steps :

(1) Starting time : 0, initial state q_0
(2) Successor case : normal Turing machine operation
(3) Limit case :
    - ⋆ `limsup` on all cells (alphabet $\{0, 1\}$)
    - ⋆ `rewind` the head at the begining of the tape
    - ⋆ `goto` limit state q_L

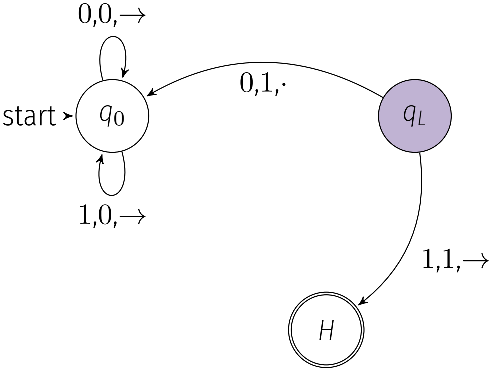

Figure – States of an ITTM

Flash algorithm on an ITTM

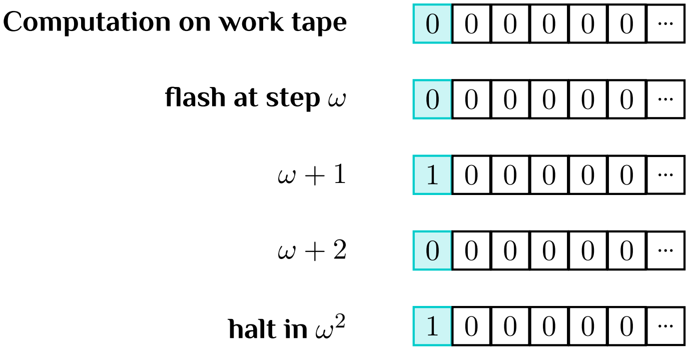

Figure – Evolution of the work tape of an ITTM

Flash algorithm on an ITTM

Computation on work tape	0	0	0	0	0	0	...
flash at step ω	0	0	0	0	0	0	...
$\omega + 1$	1	0	0	0	0	0	...
$\omega + 2$	0	0	0	0	0	0	...
halt in ω^2	1	0	0	0	0	0	...

Figure – Evolution of the work tape of an ITTM

- `input` : a real that we place on a tape
- `output` : a real that we read from a tape
- an ITTM computes a function from $\mathbb{R}$ to $\mathbb{R}$

Counting through orders

Definition (clockable ordinal)

α is a clockable ordinal if and only if $\exists$ an ITTM μ such that the computation of μ (on empty input $000\ldots$) halts in exactly α steps.

Beware : clockable ordinals do not form a segment. You can have $\alpha < \beta$ with β clockable and α not.

Theorem (Count-Through Theorem - Hamkins and Lewis, 2000)

There exists an ITTM that, given the real representation of a linear order, decides if the order is a well order in time $\alpha + \omega$ where α is the length of the w.o.i.s (well order initial segment).

Our construction

- We design an ITTM μ that takes as input a program of a Turing machine *n*
  - μ simulates the TM number *n* and checks whether it is the program for a recursive real *x*

    Turing machine simulation by ITTM - time ω
  - If yes, μ checks that *x* codes a linear order

    Simple algorithmics on ITTM, timers - time ω
  - If yes, μ counts through the order to check whether the order is a well order

    Count-Through Theorem - time $\alpha + \omega$

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- We ITTM-compute $\mu(0), \mu(1), \mu(2), \ldots$
- Let us suppose now that all such α are recursive. We conclude that our computation halts in *exactly* ω_1^{CK}
- ω_1^{CK} is *not* clockable $\Rightarrow$ contradiction !!

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- ω_1^{CK} is *not* clockable $\Rightarrow$ contradiction !!

  beware of this high level argument
- We conclude that there exists a recursive linear order the *w.o.i.s* of which has ordinal type ω_1^{CK} and we prove **Harrison's theorem**

Definition (pseudo-well ordering)

A pseudo-well ordering is a linear order that has **no** *infinite* **arithmetic** *descending chain.*

- In other terms, one cannot arithmetically differentiate a pseudo-well ordering from a well-order
- Recursive pseudo-well ordering have a *w.o.i.s* of length ω_1^{CK} (Gandy - easier proof by ITTM)
- A recursive linear order the *w.o.i.s* of which has length ω_1^{CK} is a recursive pseudo-well ordering
- Harrison proved in 1968 that there exists recursive pseudo-well orderings
- We call "Harrison real" a recursive real encoding a pseudo-well ordering

Insight : Pseudo-well orderings and Harrison's reals challenge our intuitions about linear orders and the nature of recursive sets.

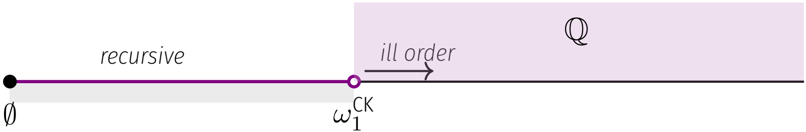

Figure – An example of a recursive pseudo-well ordering

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Some formulations of Harrison's theorem

Theorem (Original formulation)

There exists a recursive linear order of which the ordinal type of the w.o.i.s is exactly ω_1^{CK}.

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Theorem (Original formulation)

There exists a recursive linear order of which the ordinal type of the w.o.i.s is exactly ω_1^{CK}.

Theorem (First order formulation)

There exists a recursive real that codes a linear order so that any recursive ordinal is a prefix of this order.

Motivations

Original proof and classical proofs use descriptive set theory and computability results such as

$$\Sigma^1_1 \neq \Pi^1_1$$

they are rather simple but lays in the analytic hierarchy.

Some other second order proofs exist, for instance using Kleene-Brouwer order on a tree without arithmetic infinite path.

Our proof is bottom-up (”constructive”), more *elementary*.

$\longrightarrow$ We prove this now.

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Our goal is to place our proof in ACA_0

$\mathbf{ACA}_0$ stands for **Arithmetical Comprehension Axiom**

It allows comprehension (existence) of sets defined by arithmetical formulas

It contains basic Peano arithmetics RCA_0

More formally : in ACA_0, for all arithmetical formula $\varphi(n)$,
we have a set $A = \{n \mid \varphi(n)\}$

ACA_0 deals well with recursive sets and computations that involve recursive ordinals

The result $\Sigma_1^1 \neq \Pi_1^1$ states that there exists a Σ_1^1 set of reals that is not Π_1^1.

Σ_1^1 : Sets definable by an existential quantifier over all arithmetical predicates

Π_1^1 : Sets definable by a universal quantifier over all arithmetical predicates

This is a separation result in the **analytical hierarchy**

Why ACA_0 is Too Weak to Prove $\Sigma^1_1 \neq \Pi^1_1$

- ACA_0 only provides comprehension for **arithmetical formulas**
- Σ^1_1 and Π^1_1 sets require quantification over **all sets of natural numbers**, which is beyond the arithmetical realm
- the separation $\Sigma^1_1 \neq \Pi^1_1$ requires constructing a set that cannot be defined by any arithmetical formula
- ACA_0 lacks the necessary strength to express such higher-level definitions, rendering this theory too weak to contain the original proofs
- It seems that TRA is necessary, and so the proof by Harrison and others reside in ATR_0

our ITTM computations

- ◦ μ simulates the TM number *n* and checks whether it is the program for a recursive real *x*

  Turing machine simulation by ITTM - time ω
- ◦ If yes, μ checks that *x* codes a linear order

  Simple algorithmics on ITTM, timers - time ω
- ◦ If yes, μ counts through the order to check whether the order is a well order

  Count-Through Theorem - time $\alpha + \omega$

- As for Turing machines, there is an equivalence between ITTM computations and some class of formula
- If an ITTM halts in time α, then it computes a function in $\Sigma_1(L_\alpha)$. L_α is Gödel constructible set (Hamkins and Lewis 2000)

- ◦ We have used the fact that all ordinals below ω_1^{CK} are recursive – thus our ITTM enumerates them all, and make computation steps at each stage
- ◦ We need our first-order recursive version of the collapse theorem
- ◦ If an ordinal α is arithmetic, then it is also recursive. Furthermore, the transformation of the formula that caracterizes α into a program (for $\omega \cdot \alpha$) is recursive
- This result was already known in recursion theory, but we found no first order proof in litterature. We prove this using a subtle *priority argument* construction on orders.

- We remark that a Σ_1 sum of recursive ordinals is also recursive
- We use the fact that no ITTM may halt in ω_1^{CK}. The classical justification is that $\omega_1^{\mathsf{CK}}$is an *admissible* ordinals and that halting of an ITTM would contradict admissibility. This argument is too high !

◦ We replace the last argument by the combination of two facts :

ITTM computations in time β lays in $\Sigma_1(L_\beta)$
a Σ_1 sum of recursive ordinals is also recursive

$$\tau = \sum_{n \in \omega} \tau_n$$

ITTM algorithmic argument

$$\tau = \sum_{n \in \omega} \tau_n$$

- For all recursive ordinals β, there exists a TM b which produces the code x_β
- $\beta \leq \tau_b \leq \omega_1^{\mathrm{CK}} \Rightarrow \omega_1^{\mathrm{CK}} \leq \tau$
- $\forall n\ \tau_n$ is Σ_1-defined over L_{τ_n}

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- $\mu(n)$ computation is $\Sigma_1(L_{\tau_n}) \longrightarrow$ which is arithmetical
- By the collapse theorem, τ_n is recursive and $\tau = \omega_1^{\mathrm{CK}}$

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Quod erat demonstrandum

Our entire proof resides in ACA_0

Vielen Dank fr Ihre Aufmerkſamkeit

Merci pour votre attention

Thank you for liſtening