Π_4^0 conservation of Ramsey's theorem for pairs

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Joint work with Ludovic Levy Patey and Keita Yokoyama

Introduction

Motivations: Hilbert's program

Objective: Justify the use of the actual infinity in mathematics.

- Conservation: Every theorem about finite objects proved using infinite objects can be proven without them.
- Consistency: Finitary mathematics can prove that infinitary mathematics doesn't lead to a contradiction.
- Gödel (1931) : Both of these goals are unattainable.
- Partial results still possible.

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Reverse mathematics : Framework

Framework : second-order arithmetic.

- Easy distinction between finite and infinite objects.
- Allow the use of computability theory tools.
- Most of everyday mathematics is still formalizable.

Base theory RCA_0

Base theory: RCA_0

- Robinson's arithmetic Q
- Δ^0_1-comprehension (The computable sets exists)
- Σ^0_1-induction (Every set of finite cardinality is bounded)

RCA_0 is conservative over Σ_1-PA (Friedman) and Π_2 conservative over PRA (Parsons, Harrington)

The “Big Five”

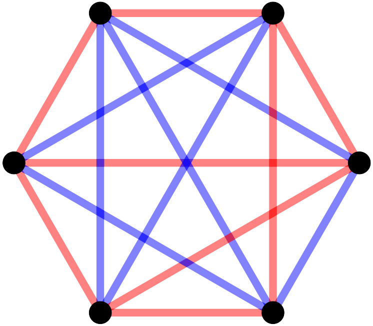
Modulo RCA_0, most theorems of ordinary mathematics are equivalent to one the following theories (from weakest to strongest):

1. RCA_0: constructive mathematics.
2. WKL_0: compactness arguments.
3. ACA_0: second-order version of Peano arithmetics.
4. ATR_0: transfinite recursion.
5. $\Pi^1_1-\mathsf{CA}$: impredicativism.

Ramsey's theorem for pairs and two colors escape this phenomenon.

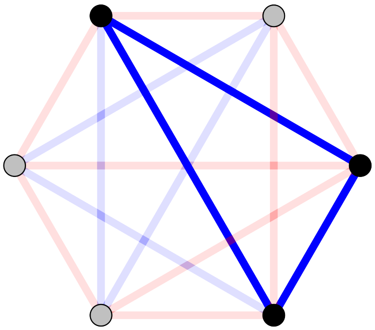
$$\begin{tikzcd}
\Pi^1_1 - \mathsf{CA}_0 \arrow[d] & \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[d] \arrow[dr] & \\
\mathsf{WKL}_0 \arrow[d] & \mathsf{RT}^2_2 \arrow[dl] \\
\mathsf{RCA}_0 &
\end{tikzcd}$$

Finite Ramsey's theorem



For every 2-coloring of the edges of K_6

Finite Ramsey's theorem



There exists some monochromatic copy of K_3

Infinite Ramsey's theorem

Let $[X]^2$ be the set of all subsets of X of cardinality 2.

Definition (Ramsey's theorem for pairs and two colors)

RT^2_2 is the statement: "For every coloring $f : [\mathbb{N}]^2 \to 2$ there is an infinite set $H \subseteq \mathbb{N}$ such that $|f([H]^2)| = 1$".

First-order consequences of RT^2_2

Facts

- $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{I}\Delta^0_2$ (Hirst)
- $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{I}\Sigma^0_2$ (Chong/Slaman/Yang)
- RT^2_2 is Π^1_1-conservative over $\mathsf{I}\Sigma^0_2 + \mathsf{RCA}_0$. (Cholak/Jockusch/Slaman)

The first-order consequences of RT^2_2 therefore lies between those of $Q + \mathsf{I}\Delta_2$ and $Q + \mathsf{I}\Sigma_2$.
It is still open whether RT^2_2 is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$

First-order consequences of RT^2_2

A $\forall\Pi^0_3$ formula is a formula of the form $(\forall X)(\forall x)(\exists y)(\forall z)\theta(X, x, y, z)$ with θ Δ^0_0.

Theorem (Patey/Yokoyama)

$\mathrm{RCA}_0 + \mathrm{RT}^2_2$ is a $\forall\Pi^0_3$-conservative extension of RCA_0.

Furthermore, the proof is formalizable in PRA, hence $PRA \vdash Con(Q + \mathrm{I}\Sigma_1) \to Con(\mathrm{RCA}_0 + \mathrm{RT}^2_2)$

Main theorem (Le Houérou/Levy Patey/Yokoyama)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is a $\forall\Pi^0_4$-conservative extension of $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$.

Proof

Outline of the proof

Theorem

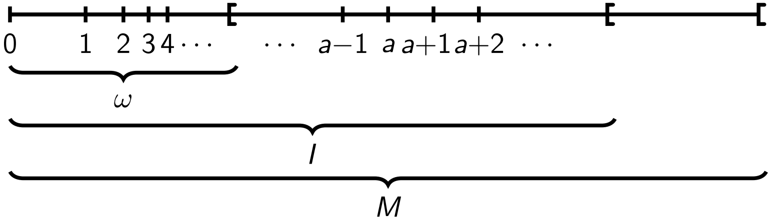
RT^2_2 *is* $\forall\Pi^0_4$ *conservative over* $\mathrm{RCA}_0 + \mathrm{I}\Delta^0_2$.

Proof:

- Assume $\mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 \not\vdash \forall X \forall x \phi(X, x)$ for $\phi(X, x) := \exists y \forall z \exists t \theta(X, x, y, z, t)$ a Σ^0_3 statement.
- By completeness, compactness and the Löwenheim-Skolem theorem, there exists $\mathcal{M} = (M, S) \models \mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 + \neg\phi(A, a)$ be a countable model with M nonstandard, and $a \in M$, $A \in S$
- From $\mathcal{M}$, build a model $\mathcal{M}' \models \mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 + \mathrm{RT}^2_2 + \neg\phi(A, a)$
- Therefore $\mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 + \mathrm{RT}^2_2 \not\vdash \forall X \forall x \phi(X)$

Cuts

An initial segment $I \subseteq M$ closed under successor is called a *cut*.



Preserving RCA_0

From a cut $I \subsetneq M$, consider the model $(I, \mathrm{Cod}(M/I))$ where

$$\mathrm{Cod}(M/I) = \{F \cap I : F \text{ finite set of } \mathcal{M}\}$$

- If I is stable by multiplication then $I \models Q$.
- $(I, \mathrm{Cod}(M/I)) \models \Delta^0_1$-comprehension.
- For $(I, \mathrm{Cod}(M/I))$ to be a model of $I\Sigma^0_1$, we want every M-finite set F of cardinality $\in I$ to not be cofinal in I. A cut verifying that is called *semi-regular*.

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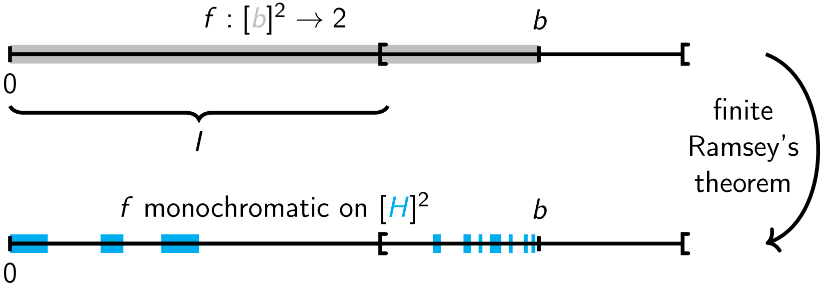
Every instance of RT^2_2 in $(I, \mathrm{Cod}(M/I))$ is obtained from a finite instance $f : [b]^2 \to 2$ that is restricted to $[I]^2$.



Problem : It may be impossible to have $H \cap I$ cofinal in I
We need a stronger version of Ramsey's theorem that put more weight on small elements.

α-largeness

Definition : α-large sets

A set $X \subseteq_{\mathtt{fin}} \mathbb{N}$ is

- ω^0-*large* if $X \neq \emptyset$.
- $\omega^{(n+1)}$-*large* if $X \setminus \min X$ is $(\omega^n \cdot \min X)$-large
- $\omega^n \cdot k$-*large* if there are k ω^n-large subsets of X

$$X_0 < X_1 < \cdots < X_{k-1}$$

where $A < B$ means that for all $a \in A$ and $b \in B$, $a < b$.

- X is $\omega^0 \cdot k$-large iff $|X| \geq k$
- X is ω^1-large iff $|X| > \min X$
- X is ω^2-large iff $X = \{\min X\} \cup X_1 \cup \cdots \cup X_{\min X}$ with each X_i ω^1-large.

Theorem: Kołodziejczyk/Yokoyama

Let X be ω^{300n}-large and $f : [X]^2 \to 2$ a coloring. There exists some ω^n-large subset Y of X such that f is homogeneous on $[Y]^2$.

Parson's theorem

If for some Δ^0_0 formula ψ we have:

$$\mathsf{RCA}_0 \vdash \forall X(X \text{ is infinite } \to (\exists F \subseteq_{\mathtt{fin}} X)\exists y \psi(y, F))$$

Then there exists some $n \in \omega$ such that:

$$\mathsf{I}\Sigma^0_1 \vdash \forall Z(Z \text{ is } \omega^n\text{-large} \to \exists F \subseteq Z \exists y < \max Z \psi(y, F))$$

Proposition

$\mathsf{RCA}_0 \vdash (\forall a)(WF(\omega^a) \to$
every infinite set contains some ω^a-large subset)

Preserving a Π^0_2 formula

Assume $(M, S) \models (\forall y)(\exists z)\theta(y, z)$.

By Δ^0_1-comprehension, let $X = \{x_0 < x_1 < \dots\}$ infinite such that $(\forall y < x_i)(\exists z < x_{i+1})\theta(y, z)$ for every i.

By overflow, let a non-standard such that $(M, S) \models \mathrm{WF}(\omega^{300^a})$
By RCA_0, let $Y \subseteq X$ be ω^{300^a}-large.

I will be defined as $\bigcup_{n\in\omega}[0, \min Y_n]$ for $Y = Y_0 \supseteq Y_1 \supseteq \dots$ with Y_i $\omega^{300^{a-i}}$-large and $\min Y_{i+1} > \min Y_i$.
Finally, $(I, \mathrm{Cod}(M/I)) \models (\forall y)(\exists z)\theta(y, z)$

Preserving a Π^0_3 formula

Assume $(M, S) \models (\forall y)(\exists z)(\forall t)\theta(y, z, t)$.
Not possible to build $X = \{x_0 < x_1 < \dots\}$ infinite such that $(\forall y < x_i)(\exists z < x_{i+1})(\forall t)\theta(y, z, t)$: this requires Σ^0_1-comprehension.

Definition: θ-apart

Two finite sets $A < B$ are θ-apart if:

$$(\forall y < \max A)(\exists z < \min B)(\forall t < \max B)\theta(y, z, t)$$

Definition : α-large(θ) sets

A set $X \subseteq_{\texttt{fin}} \mathbb{N}$ is

- ω^0-*large*(θ) if $X \neq \emptyset$.
- $\omega^{(n+1)}$-*large*(θ) if $X \setminus \min X$ is $(\omega^n \cdot \min X)$-large(θ)
- $\omega^n \cdot k$-*large*(θ) if there are k ω^n-large(θ) subsets of X that are pairwise θ-apart.

$$X_0 < X_1 < \cdots < X_{k-1}$$

For every standard n, $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + (\forall y)(\exists z)(\forall t)\theta(y, z, t)$ proves that every infinite set contain some ω^n-large(θ) set.

Proposition

Let X be $\omega^{(16^6+1)^n}$-large(θ) and $f : [X]^2 \to 2$ a coloring. There exists some ω^n-large(θ) subset Y of X such that f is homogeneous on $[Y]^2$.

References

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Quentin Le Houérou, Ludovic Levy Patey, and Keita Yokoyama.
Π^0_4 conservation of ramsey's theorem for pairs, 2024.

Appendix

Proposition: Kołodziejczyk/Yokoyama

If Y is ω^{n+1}-large and $Y = Y_0 \cup Y_1$, then there exists some $i < 2$ such that Y_i is ω^n-large.

Proposition: Le Houérou/Levy Patey/Yokoyama

For every n, there is a Δ^0_0 formula θ, a set Y that is ω^{2n-1}-large(θ) and a partition $Y = Y_0 \cup Y_1$ such that Y_0 and Y_1 are not ω^n-large(θ).