On the parameterized complexity of Δ_0 truth

Moritz Müller

joint with Yijia Chen and Keita Yokoyama

Δ_0 truth

Δ_0-TRUTH

Input: $n \in \mathbb{N}$ in unary and a Δ_0-formula $\varphi(x)$ with $n >> |\varphi|$.

Problem: is $\varphi(n)$ true?

Δ_0 truth

p-Δ_0-TRUTH

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eventually efficient ?

there exist computable $h : \mathbb{N} \to \mathbb{N}$ and efficient $\mathbb{A}$

that solves Δ_0-TRUTH on instances with $n > h(k)$.

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Examples

eventually P iff decidable in time $f(k) \cdot n^{O(1)}$
for some computable $f : \mathbb{N} \to \mathbb{N}$.

eventually L iff decidable in space $f(k) + O(\log n)$
for some computable $f : \mathbb{N} \to \mathbb{N}$.

Parameterized complexity

Parameterized problem (Q, κ)

- classical problem $Q \subseteq \{0,1\}^*$
- parameterization $\kappa : \{0,1\}^* \to \mathbb{N}$ AC^0-computable.

Write $n := |x|$ and $k := \kappa(x)$, the parameter of x.

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paraL	det. space $f(k) + O(\log(n))$	for some computable f
paraNL	nondet. space $f(k) + O(\log(n))$	for some computable f
FPT	det. time $f(k) \cdot n^{O(1)}$	for some computable f
paraNP	nondet. time $f(k) \cdot n^{O(1)}$	for some computable f

$$\text{paraL} \subseteq \text{paraNL} \subseteq \text{FPT} \subseteq \text{paraNP}$$

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Questions

1: det. space $f(k) + O(\log(n))$	for some computable f?
2: nondet. space $f(k) + O(\log(n))$	for some computable f?
3: det. time $f(k) \cdot n^{O(1)}$	for some computable f?
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Questions

1: p-Δ_0-TRUTH $\in$ paraL ?

2: p-Δ_0-TRUTH $\in$ paraNL ?

3: p-Δ_0-TRUTH $\in$ FPT ?

4: p-Δ_0-TRUTH $\in$ paraNP ?

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Theorem

1: p-Δ_0-TRUTH $\in$ paraL $\Rightarrow$ LINSPACE $\not\subseteq$ LINH

2: p-Δ_0-TRUTH $\in$ paraNL $\Rightarrow$ NLINSPACE $\not\subseteq$ LINH

3: p-Δ_0-TRUTH $\in$ FPT $\Rightarrow$ E $\not\subseteq$ LINH

4: p-Δ_0-TRUTH $\in$ paraNP $\Rightarrow$ NE $\not\subseteq$ LINH

LINH $\subseteq$ LINSPACE $\subseteq$ NLINSPACE $\subseteq$ E $\subseteq$ NE

Proof outline

Two ingredients:

An analysis of the parameterized halting problem

p-HALT
Input: $n \in \mathbb{N}$ in unary and a NTM $\mathbb{M}$.
Problem: does $\mathbb{M}$ accept the empty input in at most n steps?
Parameter: $k := |\mathbb{M}|$.

Proof outline

Two ingredients:

An analysis of the parameterized halting problem

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Input: $n \in \mathbb{N}$ in unary and a NTM $\mathbb{M}$.
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Proof outline

Two ingredients:

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Parameter: $k := |\mathbb{M}|$.

Theorem

(a) p-HALT$_= \in$ paraAC^0 iff NE $\subseteq$ LINH

(b) p-HALT$_=$ is the hardest almost tally problem in paraNP.

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A lower bound

Theorem p-Δ_0-TRUTH $\notin$ paraAC^0.

Background on parameterized halting

Conjecture

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Chen, Flum 2009/10

... iff $\mathsf{LFP}_{\mathsf{inv}}$ is not a logic for PTIME.

... iff there are no p-optimal propositional proof systems.

Open p-HALT $\notin$ paraAC^0 ?

paraAC^0

Barrington, Immerman, Straubing 1990

$(Q, \kappa) \in$ paraAC^0

iff Q is eventually FO:

there are a first-order sentence φ and a computable $h : \mathbb{N} \to \mathbb{N}$ such that

for all $x \in \{0,1\}^*$ with $|x| \geq h(\kappa(x))$: $\qquad x \in Q \iff \mathcal{S}(x) \models \varphi$.

String structure $\qquad$ Let $x = x_0 \cdots x_{n-1} \in \{0,1\}^n$ for $n > 1$.

$$\mathcal{S}(x) = \big([n], +^n, \times^n, <^n, 0, 1, \mathit{ONE}^n\big)$$

$$\mathit{ONE}^n = \big\{i \in [n] \mid x_i = 1\big\}$$
$$+^n = \big\{(i, j, k) \in [n]^3 \mid i + j = k\big\}$$
etc.

$L^{\mathsf{r}}_{\mathsf{ar}} = \{+, \times, 0, 1, <\}$ with ternary relation symbols $+, \times$.

$$p\text{-}\mathbf{HALT}_{=} \in \mathbf{paraAC}^0 \iff \mathbf{NE} \subseteq \mathbf{LINH}$$

p-**HALT**$_=$ $\in$ **paraAC**^{0} $\iff$ **NE** $\subseteq$ **LINH**

For $x \in \{0,1\}^*$ let

num(x) := the number with binary expansion $1x$.

For $Q \subseteq \{0,1\}^*$ let

un(Q) := $\{1^{num(x)} \mid x \in Q\}$.

Allender, Gore 1990

$Q \in$ LINH $\iff$ *un*$(Q) \in$ AC^0.

p-**HALT**$_=$ $\in$ **paraAC**^{0} $\iff$ **NE** $\subseteq$ **LINH**

Assume NE $\subseteq$ LINH. Consider

Q
  Input: $n \in \mathbb{N}$ in binary and a NTM $\mathbb{M}$.
  Problem: does $\mathbb{M}$ accept the empty input in exactly n steps?

Then $Q \in$ LINH. Hence AC^0 contains

$$un(Q) \;=\; \left\{1^{num(\langle n,\mathbb{M}\rangle)} \mid \mathbb{M} \text{ accepts the empty input in exactly } n \text{ steps}\right\}$$

Then p-HALT$_=$ $\in$ paraAC^0 because

$$\langle 1^n, \mathbb{M}\rangle \mapsto 1^{num(\langle n,\mathbb{M}\rangle)}$$

is a suitable reduction to $un(Q)$.

p-HALT$_=$ $\in$ paraAC^0 $\iff$ NE $\subseteq$ LINH

Assume p-HALT$_=$ $\in$ paraAC^0. Let $Q \in$ NE.

Want: *un*$(Q) \in$ AC^0.

Choose $c \in \mathbb{N}$ and an NTM $\mathbb{M}$ for Q in time *num*$(x)^c - 2|x|$

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Define $\mathbb{M}^*$ on the empty input:

1: guess $y \in \{0,1\}^*$ in exactly $2|y|$ steps.

2: run $\mathbb{M}$ on y.

3: if $\mathbb{M}$ rejects, reject.

4: make dummy steps to complete *num*$(y)^c$ steps.

5: accept.

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5: accept.

Then

$$1^{num(x)} \in un(Q) \iff \mathbb{M}^* \text{ accepts in exactly } num(x)^c + 1 \text{ steps.}$$

Since p-HALT$_=$ $\in$ paraAC^0 and $\mathbb{M}^*$ is a fixed machine: r.h.s. is AC^0. $\square$

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$r : \{0,1\}^* \to \{0,1\}^*$ is an eventually definable reduction (Q, κ) to (Q', κ') if:

(a) $|r(x)| \geq |x|^{\Omega(1)}$.

(b) $\kappa' \circ r \leq f \circ \kappa$ for some computable $f : \mathbb{N} \to \mathbb{N}$.

(c) $x \in Q \iff r(x) \in Q'$.

(d) exist computable h, interpretation I st:

$$\mathcal{S}(x)^I \cong \mathcal{S}(r(x)).$$

for every $x \in \{0,1\}^*$ with $|x| \geq h(\kappa(x))$.

Lemma

This reducibility is transitive and preserves paraAC^0.

p-HALT$_=$ is the hardest almost tally problem in paraNP.

(Q, κ) is almost tally if for some computable $f : \mathbb{N} \to \mathbb{N}$:

- $Q \subseteq \big\{\langle 1^n, x\rangle \mid n \in \mathbb{N}, x \in \{0,1\}^*\big\}$
- $|x| \le f\big(\kappa(\langle 1^n, x\rangle)\big)$.

- p-HALT$_=$, p-HALT, p-Δ_0-TRUTH are almost tally.

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Lemma

For every almost tally problem in paraNP there is an eventually definable reduction to p-HALT$_=$.

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Lemma

For every almost tally problem in paraNP there is an eventually definable reduction to p-HALT$_=$.

Corollary

NE $\subseteq$ LINH iff every almost tally problem in paraNP is in paraAC^0.

p-Δ_0-TRUTH $\notin$ paraAC^0.

Otherwise model-checking arithmetic is in paraAC^0:

p-MC($L^{\mathsf{r}}_{\mathsf{ar}}$)
Input: $n \geq 2$ in unary and an $L^{\mathsf{r}}_{\mathsf{ar}}$-sentence φ.
Problem: $n \models \varphi$?
Parameter: $k := |\varphi|$.

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Thus there are a sentence *sat* and a computable $h : \mathbb{N} \to \mathbb{N}$ such that

$$n \models \varphi \iff \mathcal{S}(\langle 1^n, \varphi \rangle) \models \mathit{sat}$$

for all $n \geq h(\mathit{num}(\varphi))$.

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Construct a $L^{\mathsf{r}}_{\mathsf{ar}}$-formula *true*($x$) such that

$$n \models \varphi \iff n \models \textit{true}(\textit{num}(\varphi))$$

for all $n \geq h(\textit{num}(\varphi))$.

p-Δ_0-**TRUTH** $\notin$ **paraAC**^{0}**.**

Then

$$\mathbb{N} \models \text{“}h(\mathit{num}(\varphi)) \leq y\text{”} \;\; \rightarrow \;\; \big(\varphi^{<y} \leftrightarrow \mathit{true}^{<y}(\mathit{num}(\varphi)).\big)$$

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Then

$$\mathbb{N} \models \text{“}h(\mathit{num}(\varphi)) \le y\text{”} \;\; \to \;\; \big(\varphi^{<y} \leftrightarrow \mathit{true}^{<y}(\mathit{num}(\varphi)).\big)$$

Let M be nonstandard and $a \in M \setminus \mathbb{N}$. Then

$$M \models \varphi^{<a} \leftrightarrow \mathit{true}^{<a}(\mathit{num}(\varphi))$$

for all $L^{\mathsf{r}}_{\mathsf{ar}}$-sentences φ.

Contradiction by standard diagonalization. $\square$

Upper bounds

Theorem

If p-Δ_0-TRUTH $\in$ paraNP, then NE $\not\subseteq$ LINH.

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If p-Δ_0-TRUTH $\in$ paraNL, then NLINSPACE $\not\subseteq$ LINH.

If p-Δ_0-TRUTH $\in$ paraL, then LINSPACE $\not\subseteq$ LINH.

$$\text{LINH} \subseteq \text{LINSPACE} \subseteq \text{NLINSPACE} \subseteq \text{E} \subseteq \text{NE}$$

The MRDP theorem

Question Does $I\Delta_0$ prove MRDP?

for every $\varphi(\bar{x}) \in \Delta_0$ there are terms $p(\bar{x},\bar{y}), q(\bar{x},\bar{y})$ st $I\Delta_0$ proves

$$\varphi(\bar{x}) \;\leftrightarrow\; \exists \bar{y}\; p(\bar{x},\bar{y}) = q(\bar{x},\bar{y}).$$

Wilkie 1980 Then NP = coNP.

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Conjecture $I\Delta_0$ proves MRDP for small numbers:

for every $\varphi(x) \in \Delta_0$ there are terms $p(x, \bar{y}), q(x, \bar{y})$ st $I\Delta_0$ proves

$$2^x \leq z \ \rightarrow \ \big(\varphi(x) \ \leftrightarrow \ \exists \bar{y} \ p(x, \bar{y}) = q(x, \bar{y})\big).$$

Intuitively Much weaker than $I\Delta_0$-provability.

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Then NE $\not\subseteq$ LINH.

The MRDP theorem

Theorem Let T be a true, c.e. Π_1-theory.

If T proves MRDP for small numbers, then NE $\not\subseteq$ LINH.

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Theorem Let T be a true, c.e. Π_1-theory.

If T proves MRDP for small numbers, then NE $\not\subseteq$ LINH.

Proof Parikh's Theorem implies:

for every $\varphi(x)$ there are $p(x,\bar{y}), q(x,\bar{y}), r(x,z)$ such that T proves

$$2^x = z \;\rightarrow\; \big(\varphi(x) \;\leftrightarrow\; \exists \bar{y}{<}r(x,z)\; p(x,\bar{y}) = q(x,\bar{y})\big)$$

Solve p-Δ_0-TRUTH on input $\langle 1^n, \varphi(x)\rangle$:

1: compute p, q, r from φ as above. // since T is c.e..
2: guess $\bar{m} < r(n, 2^n)$. // length $O(|\bar{y}| \cdot |r| \cdot n)$
3: check $p(n,\bar{m}) = q(n,\bar{m})$ // time poly $|p| \cdot |q| \cdot |r| \cdot n$

Hence p-Δ_0-TRUTH $\in$ paraNP, so NE $\not\subseteq$ LINH. □

Relaxing uniformity

XAC^0 contains (Q,κ) if every slice is in AC^0.

XAC^0_{const} . . . and depth independent of the slice.

$$\mathsf{paraAC}^0 \subseteq \mathsf{XAC}^0_{const} \subseteq \mathsf{XAC}^0$$

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$$\mathsf{paraAC}^0 \subseteq \mathsf{XAC}^0_{const} \subseteq \mathsf{XAC}^0$$

Theorem

p-HALT $\in \mathsf{XAC}^0_{const}$.

p-HALT$_=$ $\in \mathsf{XAC}^0_{const}$ iff NE $\subseteq$ LINH.

p-Δ_0-TRUTH $\in \mathsf{XAC}^0_{const}$ iff LINH collapses.

Problem comparison

$$p\text{-HALT}_=$$

$$\uparrow$$

$$p\text{-HALT}$$

Corollary

p-HALT$_=$ $\not\equiv$ p-HALT unless NE $\subseteq$ LINH.

Problem comparison

$$
\begin{array}{ccc}
p\text{-HALT}_{=} & \not\equiv & p\text{-}\Delta_0\text{-TRUTH} \\
\uparrow & & \\
p\text{-HALT} & &
\end{array}
$$

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$$
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 & p\text{-SPEC} & \\
\nearrow & & \nwarrow \\
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p-SPEC
Input: $n \in \mathbb{N}$ in unary and a first-order sentence φ.
Problem: does φ have a model of size n?
Parameter: $k := |\varphi|$.