

Topological Characterization of Gurevich Modelization of Sequential Algorithms

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Abstract. Sequential algorithms over countable data structures are modeled by Gurevich’s Abstract State Machines as the iteration of a functional which goes from a finite product of function spaces into itself. ASM functionals have the following properties for some k : (1) _{k} they modify their argument on at most k points, (2) _{k} their modulus of continuity is k -bounded. We show that these properties characterize ASM functionals and that property (2) _{k^2} then holds with a modulus of continuity given by ground terms. This $k \mapsto k^2$ blow-up is optimal. We also give a topological interpretation of property (2) in terms of uniform continuity. The effective version also holds but is harder to prove and involves a super-exponential blow-up due to the use of Ramsey’s theorem.

Keywords Specification; Abstract state machines; Computability.

1 Introduction

1.1 Roadmap

We consider deterministic sequential algorithms working with countable data structures. Here sequential means discrete time plus uniformly bounded action (as opposed to “wild” parallel action). This a priori quite informal notion of algorithm has been convincingly modeled around 1984 by Yuri Gurevich with Abstract State Machines (aka Evolving Algebras), see [6,7,8]. A (deterministic) ASM consists of the following data.

- A multisort logical functional structure $\mathfrak{M} = (M_1, \dots, M_n; \mathbf{f}_1, \dots, \mathbf{f}_p)$ on a so-called static functional vocabulary (f_1, \dots, f_p) . This static vocabulary is typed: types are of the form M_j or $M_{j_1} \times \dots \times M_{j_q} \rightarrow M_r$. The interpretations \mathbf{f}_i ’s respect the types of the f_i ’s.
- A so-called dynamic functional vocabulary which is similarly typed.
- An ASM program consisting of assignments $f(t_1, \dots, t_\ell) := u$ where f is a dynamic symbol and the t_i ’s and u are ground terms constructed with both vocabularies, conditionals and finite blocks of such instructions.

A state of an ASM is an interpretation of the dynamic vocabulary. A run of an ASM is a sequence of states obtained from the first one (the “initial” state) by iterated application of the ASM program. Thus,

- The base set of the structure and the interpretations of the static symbols are fixed. They represent the framework of the algorithm and the primitive operations “given for free”, let them be computable or not.
- The interpretations of the dynamic symbols vary from state to state. They represent the dynamic environment of the algorithm.

In this way, an algorithm appears as a static framework plus a functional mapping the interpretations of the dynamic symbols in some state into the interpretations in the next state (obtained by application of the ASM program). This functional is of the form $\Psi : T \rightarrow T$ with $T = \prod_{i \in I} \left(\left(\prod_{j \in J_i} M_j \right) \rightarrow M_{\ell_i} \right)$ where I and the J_i 's are finite multisubsets of $\{1, \dots, n\}$ (recall n is the number of sorts of \mathfrak{M}), and the ℓ_i 's are in $\{1, \dots, n\}$.

Let k be the total number of occurrences of dynamic symbols in the terms of the ASM program. The ASM functional has the following properties:

- (1) _{k} It modifies its argument $\mathbf{f} = (f_1, \dots, f_q)$ on a set $X_{\mathbf{f}}$ of at most k points.
- (2) _{k} $X_{\mathbf{f}}$ and the new values of \mathbf{f} on $X_{\mathbf{f}}$ depend only on the values of \mathbf{f} on a set $Y_{\mathbf{f}}$ of at most k points. I.e., the modulus of continuity is k -bounded.

In this work, we show (Theorem 6.2) that any functional Ψ satisfying properties (1) _{k} and (2) _{k} is an ASM functional. More precisely, one can find some ASM static framework on the multisort domain (M_1, \dots, M_n) such that properties (1) _{k^2} and (2) _{k^2} hold in an ASM-like way: the k^2 points of $X_{\mathbf{f}}$ and the new values of \mathbf{f} on them are given by ground terms built with \mathbf{f} and the ASM static framework, cf. Theorem 3.7, Corollaries 3.8, 3.9. This $k \mapsto k^2$ blow-up cannot be improved (cf. Proposition 3.3).

This characterization can also be seen as a topological one using a particular transitive uniformity compatible with the usual product topology on function spaces. ASM functionals $\Psi : T \rightarrow T$ are exactly those for which, for some k , there exist $2k$ uniformly continuous functionals $A_i, B_i : T \rightarrow M_j$ such that $\Psi(f)(x) = B_i(f)$ if $x = A_i(f)$ for some $i = 1, \dots, k$ and $\Psi(f)(x) = f(x)$ otherwise. A variant topological characterization uses a notion of “linear” uniform continuity of Ψ . All these results hold in an effective version (Theorems 5.4, 6.2) with functionals associated to ASMs with computable static functions. However, the blow-up $k \mapsto k^2$ becomes a super-exponential blow-up since we have to use a completely different proof involving Ramsey’s theorem about k -colorings.

As a perspective work (in progress), these results seem to also apply to parallel algorithms, cf. Blass & Gurevich [1,2].

1.2 Conventions and notations

Convention. The notation $f : A \rightarrow B$ is used for total functions from A to B whereas $u : A \dashrightarrow B$ is used for partial functions. In practice, we shall have $A = \mathbb{N}$ or $A = \mathbb{N} \cup \{\top\}$ and $B = \mathbb{N}$. We shall use letters Φ, Ψ as respective names for functionals $\Phi : B^A \rightarrow B$ and $\Psi : B^A \rightarrow B^A$.

Definition 1.1. 1. If $\xi, \eta : A \rightarrow B$ are partial functions, we let $\xi \oplus \eta : A \rightarrow B$ be the partial function such that $\text{Domain}(\xi \oplus \eta) = \text{Domain}(\xi) \cup \text{Domain}(\eta)$, $\xi \oplus \eta$ extends η and $(\xi \oplus \eta)(x)$ is equal to $\xi(x)$ if $x \in \text{Domain}(\xi) \setminus \text{Domain}(\eta)$.
2. We write (a, b) for the function with domain $\{a\}$ which maps a onto b and $((a_1, b_1), \dots, (a_n, b_n))$ in place of $(a_1, b_1) \oplus \dots \oplus (a_n, b_n)$.

Definition 1.2. 1. $\mathfrak{F}^{A \rightarrow B}$ (resp. $\mathfrak{F}_k^{A \rightarrow B}$) is the family of all partial functions $A \rightarrow B$ with finitely many (resp. at most k) points in their domains. If $u \in \mathfrak{F}^{A \rightarrow B}$, we let $[u]$ be the set of total functions $A \rightarrow B$ which extend u . If a_1, \dots, a_n are distinct, we let $((a_1, b_1), \dots, (a_n, b_n))$ be the partial function $u \in \mathfrak{F}_n^{A \rightarrow B}$ with domain $\{a_1, \dots, a_n\}$ such that $u(a_i) = b_i$ for $i = 1, \dots, n$.
2. We write $\mathfrak{F}, \mathfrak{F}_k$ when $A = B = \mathbb{N}$ and $\mathfrak{F}^\top, \mathfrak{F}_k^\top$ when $A = \mathbb{N} \cup \{\top\}, B = \mathbb{N}$.

Definition 1.3 (Curryfication). 1. If $\Psi : B^A \rightarrow B^A$ is a total functional, we denote by $\partial\Psi$ the total functional $\partial\Psi : (B^A \times A) \rightarrow B$ such that $(\partial\Psi)(f, a) = \Psi(f)(x)$ for all $f \in B^A$ and $a \in A$.
2. We identify A with $\{\top\} \rightarrow A$ and $B^A \times A$ with $B^{A \cup \{\top\}}$, and $\partial\Psi$ with a total functional $\partial\Psi : B^{A \cup \{\top\}} \rightarrow B$.

2 Continuity and finite information

First, we reformulate continuity of functionals in terms of how much information is used. The core of the paper is to replace “finite” by “at most k ”.

Definition 2.1. We use Definitions 1.2, 1.3. 1. A covering π of B^A is a subfamily of $\mathfrak{F}^{A \rightarrow B}$ such that $B^A = \bigcup_{u \in \pi} [u]$. A covering π is deterministic if the $[u]$'s, $u \in \pi$, are pairwise disjoint, hence form a partition of B^A .
2. Let $\Phi : B^A \rightarrow B$ be a total functional. A [deterministic] Φ -family is a [deterministic] covering π such that Φ is constant on $[u]$ for every $u \in \pi$. We say that Φ uses [deterministic] finite information if there is a [deterministic] Φ -family.
2. Let $\Psi : B^A \rightarrow B^A$ be a total functional. A [deterministic] Ψ -family is a [deterministic] $\partial\Psi$ -family. Ψ uses [deterministic] finite information if so does $\partial\Psi$.

Proposition 2.2. A total functional $\Psi : B^A \rightarrow B^A$ uses finite information if and only if so does, for each $a \in A$, the $B^A \rightarrow B$ functional $f \mapsto \Psi(f)(a)$.

Using finite information is a mere reformulation of continuity.

Proposition 2.3. Let A, B be countable (in practice, A is \mathbb{N} or $\mathbb{N} \cup \{\top\}$, B is \mathbb{N} or $\{0, 1\}$). On B we consider the discrete topology and on B^A we consider the usual product topology for which the family of sets $[u]$, $u \in \mathfrak{F}^{A \rightarrow B}$, constitutes a basis of open sets. For a total functional $B^A \rightarrow B$ or $B^A \rightarrow B^A$, the following conditions are equivalent.

- (i) to be continuous.
- (ii) to use finite information.
- (iii) to use deterministic finite information.

3 Uniformly bounded information

3.1 From finite to uniformly bounded information

Definition 3.1. Let $\Phi : B^A \rightarrow B$ be a total functional, $k \in \mathbb{N}$.

1. A [deterministic] k -covering π of B^A is a [deterministic] covering (cf. Definition 2.1) included in $\mathfrak{F}_k^{A \rightarrow B}$. A covering is uniformly bounded if it is a k -covering for some $k \in \mathbb{N}$.

2. A (Φ, k) -family [(Φ , bounded)-family] is a Φ -family which is a k -covering [uniformly bounded covering]. Φ uses [deterministic] k -bounded information if there exists a [deterministic] (Φ, k) -family. Φ uses [deterministic] uniformly bounded information if it uses [deterministic] uniformly bounded information.

Example 3.2. A trivial example of a deterministic k -covering of $\mathbb{N}^{\mathbb{N}}$ is that of all functions having some fixed domain $\{s_1, \dots, s_k\}$ with k elements. A less trivial example, with $k = 2$, is the family of all functions u with graph of the form $\{(a, y), (\theta(y), z)\}$ (in other words, $\text{Domain}(u) = \{a, \theta(u(a))\}$) where $a \in \mathbb{N}$ and $\theta : \mathbb{N} \rightarrow \mathbb{N}$ are fixed but y, z vary in \mathbb{N} .

The analog of equivalence (ii) \Leftrightarrow (iii) of Proposition 2.3 fails with respect to bounded information. The next result shows that, in general, the existence of a (Φ, k) -family does not imply that of a deterministic (Φ, k) -family. A quadratic blow-up is unavoidable (it is also sufficient, cf. Theorem 3.7).

Proposition 3.3. *There exists a total functional $\Phi : \mathbb{N}^{\mathbb{N}} \rightarrow \{0, 1\}$ which uses k -bounded information and deterministic k^2 -bounded information but cannot use only deterministic $(k^2 - 1)$ -bounded information.*

Proof. Let $\Phi(f) = 0$ if f is 0 on some block $ik, \dots, ik + k - 1$ and 1 otherwise.

3.2 Constructors

The next Definition and Lemma extend the idea of Example 3.2 to get simply definable deterministic bounded coverings of B^A .

Definition 3.4. 1, A constructor is a tuple

$$\Theta = (s_1, \dots, s_{\ell_0}, \theta_1^{(1)}, \dots, \theta_{\ell_1}^{(1)}, \dots, \theta_1^{(p)}, \dots, \theta_{\ell_p}^{(p)}, \omega)$$

where, letting $\lambda(i) = \sum_{m=0}^{m=i-1} \ell_m$ for $i = 1, \dots, p$, and $\lambda = \sum_{i=1, \dots, p} \lambda(i)$,

- (i) $p, \ell_0, \ell_1, \dots, \ell_p \in \mathbb{N}$ and $\ell_i = 0 \Rightarrow \ell_{i+1} = 0$ for $i = 0, \dots, p - 1$,
- (ii) the s_j 's are elements of A (the seeds of the constructor),
- (iii) the $\theta_j^{(i)}$'s are total functions $\theta_j^{(i)} : B^{\lambda(i)} \rightarrow A$.
- (iv) ω is a total function $\omega : B^\lambda \rightarrow B$.

2. To Θ we associate partial functionals, $\alpha_j^{(i)} : (A \multimap B) \multimap A$, $\beta_j^{(i)} : (A \multimap B) \multimap B$, $\Omega_\Theta : (A \multimap B) \multimap B$, for $0 \leq i \leq p$ and $1 \leq j \leq \ell_i$, inductively defined as follows: for $j \in \{1, \dots, \ell_i\}$, $\xi \in \mathfrak{F}^{A \multimap B}$,

$$\begin{aligned} \alpha_j^{(0)}(\xi) &= s_j & \beta_j^{(i)}(\xi) &= \xi(\alpha_j^{(0)}(\xi)) \text{ for } i = 0, \dots, p \\ \alpha_j^{(i)}(\xi) &= \theta_j^{(i)} \left((\beta_{j'}^{(i')}(\xi))_{\substack{j'=0, \dots, i-1 \\ j'=1, \dots, \ell_{i'}}} \right) & \text{ for } i = 1, \dots, p \\ \Omega_\Theta(\xi) &= \omega \left((\beta_j^{(i)}(\xi))_{\substack{i=0, \dots, p \\ j=1, \dots, \ell_i}} \right) \end{aligned}$$

3. $u \in \mathfrak{F}^{A \multimap B}$ is Θ -good if Ω_Θ is defined, i.e. if all $\alpha_j^{(i)}(u)$'s are defined and $\text{Domain}(u)$ contains $\mathfrak{s}_\Theta(u) = \bigcup_{i=0, \dots, p} \{\alpha_j^{(i)}(\xi) \mid 1 \leq j \leq \ell_i\}$.

The following result is straightforward.

Proposition 3.5. *Let Θ be a constructor as in Definition 3.4. A partial function u is minimal among Θ -good functions relative to function extension if and only if $\text{Domain}(u) = \mathfrak{s}_\Theta(u)$.*

Lemma 3.6. *The family π_Θ of minimal Θ -good functions is a deterministic $(\sum_{m=0}^{m=p} \ell_m)$ -covering.*

3.3 Main theorem on uniformly bounded information

We can now prove that Proposition 3.3 is optimal: a quadratic blow-up is sufficient for determinization. This is the first main theorem of the paper.

Main Theorem 3.7. *If the total functional $\Phi : B^A \rightarrow B$ uses k -bounded information then it uses deterministic k^2 -bounded information and there exists a constructor Θ as in Definition 3.4 such that $\Phi = \Omega_\Theta \upharpoonright B^A$ and π_Θ is a deterministic (Φ, k^2) -family and*

- (a) if $k = 0$ then Θ is reduced to (ω) and π_Θ is the singleton family consisting of the function with empty domain,
- (b) if $k \geq 1$ then $p = k - 1$ and $\ell_i = 2(k - i) - 1$ for $i = 0, \dots, p$.

As immediate corollaries, we get.

Main Corollary 3.8. *The following conditions are equivalent.*

- 1. Φ uses uniformly bounded information.
- 2. Φ uses deterministic uniformly bounded information.
- 3. There exists a constructor Θ as in Definition 3.4 such that $\Phi = \Omega_\Theta \upharpoonright B^A$.

Main Corollary 3.9. *If a total functional $\Phi : B^A \rightarrow B$ uses k -bounded information then there exists a total functional $\Psi : B^A \rightarrow \mathcal{F}_{k^2}^{A \multimap B}$ such that,*

- (a) For all $f \in B^A$, Ψ and Φ are constant on $[\Psi(f)]$ and $f \in [\Psi(f)]$. Thus, the range of Ψ is a deterministic uniformly bounded Ψ -family and Φ -family.
- (b) Ψ is uniformly continuous.

Proof. Set $\Psi(f) = f \upharpoonright \mathfrak{s}_\Theta(f) = f \upharpoonright \bigcup_{i=0, \dots, p} \{\alpha_j^{(i)}(f) \mid 1 \leq j \leq \ell_i\}$ where the $\alpha_j^{(i)}$'s are constructed as in Definition 3.4. \square

3.4 Proof of Theorem 3.7

It is easy to see that the proof given for Proposition 3.3 does not preserve boundedness, a different argument is required to prove Theorem 3.7. We shall argue by induction on $k \geq 1$, the case $k = 0$ being special and trivial.

First, we introduce convenient notation and prove some auxiliary results.

Definition 3.10. *Suppose $\Phi : B^A \rightarrow B$ is a total functional. We let X_Φ [$X_{\Phi,k}$] be the family of partial functions $u : \mathbb{N} \rightarrow \mathbb{N}$ such that Φ is constant on $[u]$ and $\text{Domain}(u)$ is finite [has at most k elements]. We let $\varphi_\Phi : X_\Phi \rightarrow \mathbb{N}$ be such that, for $u \in X_\Phi$, $\varphi_\Phi(u)$ is the common value of all $\Phi(f)$'s for $f \in [u]$.*

Proposition 3.11. *If Φ uses k -bounded information ($k \geq 1$) then $X_{\Phi,k}$ is infinite.*

Proposition 3.12. *If $u, v \in X_\Phi$ agree on $\text{Domain}(u) \cap \text{Domain}(v)$ (in particular, if v extends u or if they have disjoint domains) then $\varphi_\Phi(u) = \varphi_\Phi(v)$.*

Proof. Consider $f \in \mathbb{N}^{\mathbb{N}}$ which extends both u and v . Then $\Phi(f) = \varphi_\Phi(u)$ and $\Phi(f) = \varphi_\Phi(v)$ hence $\varphi_\Phi(u) = \varphi_\Phi(v)$. \square

The following result will be used for effectivization to prove Lemma 5.3.

Proposition 3.13. *Suppose Φ uses k -bounded information, u_1, \dots, u_{k+1} in $X_{\Phi,k}$, $D \subset A$, $v \in B^D$ are such that $D = \text{Domain}(u_i) \cap \text{Domain}(u_j)$ for all $i < j$ and $u_i \upharpoonright D = v$ for all i 's. Then Φ is constant on $[v]$.*

Proof. Applying Proposition 3.12, we see that $\varphi_\Phi(u_1) = \dots = \varphi_\Phi(u_{k+1})$. Let $f \in [v]$. Then $f \in [u]$ for some $u \in X_{\Phi,k}$. Since $\text{Domain}(u)$ has at most k elements there exists i such that $\text{Domain}(u) \cap \text{Domain}(u_i) \subseteq D$. Since u, u_i agree with v on D , Proposition 3.12 insures $\varphi_\Phi(u) = \varphi_\Phi(u_i) = \varphi_\Phi(u_1)$. Thus, $\Phi(f) = \varphi_\Phi(u_1)$, so Φ is constant on $[v]$.

The next Proposition gives the case $k = 1$ of Theorem 3.7.

Proposition 3.14. *If $\Phi : B^A \rightarrow B$ is a total functional using 1-bounded information then for every $s \in \bigcup_{u \in X_{\Phi,1}} \text{Domain}(u)$, the functional Φ is constant on $[u]$ for every u with domain $\{s\}$ (in other words, letting $\Theta = (s)$, π_Θ is a $(\Phi, 1)$ -family).*

Proposition 3.15. *Let $\Phi : B^A \rightarrow B$ be a k -bounded total functional. If Φ is not constant then there exists a set S with at most $2k - 1$ elements such that the domain of every partial function $u \in X_\Phi$ meets S .*

Definition 3.16. *Suppose $\Phi : B^A \rightarrow B$ is a total functional and $S \subset A$ is finite and $w \in B^S$. We let $\Phi^{(w)} : B^A \rightarrow B$ be the total functional such that $\Phi^{(w)}(f) = \Phi(f \oplus w)$ for every $f \in B^A$.*

Proposition 3.17. *Let $\Phi : B^A \rightarrow B$ be a total functional using k -bounded information. Suppose that there is a non empty set S such that the domain of every function $u \in X_{\Phi,k}$ meets S . Then, for every function $w \in \mathbb{N}^S$, the total functional $\Phi^{(w)} : B^A \rightarrow B$ is $(k - 1)$ -bounded.*

Proposition 3.18. *Let S be a finite non empty subset of A and , for every $w \in B^S$, let $\Phi^{(w)} : B^A \rightarrow B$ be a total functional and*

$$\Theta^{(w)} = (s_1^{(w)}, \dots, s_{\ell_0}^{(w)}, \theta_1^{(w,1)}, \dots, \theta_{\ell_1}^{(w,1)}, \dots, \theta_1^{(w,p_w)}, \dots, \theta_{\ell_{p_w}}^{(w,p_w)}, \omega^{(w)})$$

be a constructor as in Definition 3.4 such that $\Phi^{(w)}$ is constant on $[u]$ for every Θ_w - good partial function u . Let $\Phi : B^A \rightarrow B$ be the unique total functional such that, for every $f \in B^A$, $\Phi(f) = \Phi^{(f|S)}(f)$. Suppose that there exists some $K \in \mathbb{N}$ which majorizes all $p^{(w)}$'s and $\ell_i^{(w)}$'s, for $w \in B^S$ and $0 \leq i \leq \ell_{p^{(w)}}$. Then there exists a constructor $\Theta = (s_1, \dots, s_{\ell_0}, \theta_1^{(1)}, \dots, \theta_{\ell_1}^{(1)}, \dots, \theta_1^{(p)}, \dots, \theta_{\ell_p}^{(p)}, \omega)$ as in Definition 3.4 such that $\Phi = \Omega_{\Theta} \upharpoonright B^A$ and

- (i) $\{s_1, \dots, s_{\ell_0}\} = S$,
- (ii) $p = 1 + \max\{p_w \mid w \in B^S\}$, $\ell_i = \max\{\ell_{i-1}^{(w)} \mid p_w \geq i - 1\}$ for $i = 1, \dots, p$.
- (iii) Φ is constant on $[u]$ for every Θ -good partial function $u : A \rightarrow B$.

Proof. Let p , the s_j 's and ℓ_i 's be as given by clause (ii). First, we define the total functions $\theta_j^{(i)}$'s, $i = 1, \dots, p$. Given $\sum_{i=0}^{p-1} \ell_i$ elements $b_{\ell}^{(i)}$ of \mathbb{N} , for $i = 1, \dots, p$, we get a tuple which can be an input for the $\theta_j^{(i)}$'s:

$$D^{(i)} = (b_1^{(0)}, \dots, b_{\ell_0}^{(0)}, \dots, b_1^{(i-1)}, \dots, b_{\ell_{i-1}}^{(i-1)})$$

For $w \in B^S$ and $1 \leq i \leq p_w$, we consider the tuple obtained from $D^{(i)}$ by removing the first blocks of b 's and trimming blocks of length ℓ_i to length $\ell_{i-1}^{(w)}$, so as to get a tuple which can be an input for the $\theta_j^{(w,i-1)}$'s:

$$E^{(w,i-1)} = (b_1^{(1)}, \dots, b_{\ell_0}^{(1)}, \dots, b_1^{(i-1)}, \dots, b_{\ell_{i-1}^{(w)}}^{(i-1)})$$

We are now ready to define the $\theta_j^{(i)}$'s. Let $w \in B^S$ be such that $w(s_1) = b_1^{(0)}, \dots, w(s_{\ell_0}) = b_{\ell_0}^{(0)}$. Then, for $i = 1, \dots, p$ and $1 \leq j \leq \ell_i$, we set

$$(*) \quad \theta_j^{(i)}(D^{(i)}) = \begin{cases} s_j^{(w)} & \text{if } i = 1, j \leq \ell_0^{(w)} \\ \theta_j^{(w,i-1)}(E^{(i-1)}) & \text{if } 1 \leq i - 1 \leq p_w, j \leq \ell_{i-1}^{(w)} \\ b_1^{(0)} & \text{if } i - 1 > p_w \text{ or } j > \ell_{i-1}^{(w)} \end{cases}$$

Now that the $\theta_j^{(i)}$'s are defined, let us prove that Φ is constant on $[u]$ for every $u \in \pi_{\Theta}$. Let $f, g \in [u]$, we have to prove that $\Phi(f) = \Phi(g)$. Since the seeds of Θ are $S = \{s_1, \dots, s_{\ell_0}\}$, the domain of u contains S . Let w be the restriction of u to S . The hypothesis of the Proposition insures that Φ and $\Phi^{(w)}$ coincide on $[w]$. In particular, they coincide on $[u]$ hence $\Phi(f) = \Phi^{(w)}(f)$ and $\Phi(g) = \Phi^{(w)}(g)$. Since $u \in \pi_{\Theta}$, the $\alpha_j^{(i)}(u)$'s for $i = 0, \dots, p$ and $1 \leq j \leq \ell_i$, are all defined and belong to $\text{Domain}(u)$. Now, arguing with $\Phi^{(w)}$ and the $\alpha_j^{(w,i)}$'s, we show, by induction on

$i = 0, \dots, p_w$ that $\alpha_j^{(w,i)}(u) = \alpha_j^{(i+1)}(u)$ for all $j = 1, \dots, \ell_i^{(w)}$.

Initial case $i = 0$. Using the definitions of $\alpha_j^{(w,0)}(u)$ and $\theta_1^{(w)}$, we get

$$\begin{aligned} \alpha_j^{(w,0)}(u) &= s_j^{(w)} = \theta_j^{(1)}(D^{(1)}) = \theta_j^{(1)}(b_1^{(0)}, \dots, b_{\ell_0}^{(0)}) \\ &= \theta_j^{(1)}(w(s_1), \dots, w(s_{\ell_0})) \\ &= \alpha_j^{(1)}(w) = \alpha_j^{(1)}(u) \quad \text{since } u \text{ extends } w. \end{aligned}$$

Inductive step: from $0 \leq i-1 < p_w$ to i . We have

$$\alpha_j^{(w,i)}(u) = \theta_j^{(w,i)} \left(\beta_1^{(w,0)}(u), \dots, \beta_{\ell_0^{(w)}}^{(w,0)}(u), \dots, \beta_1^{(w,i-1)}(u), \dots, \beta_{\ell_{i-1}^{(w)}}^{(w,i-1)}(u) \right)$$

By induction hypothesis, for $i' < i$, $\alpha_j^{(w,i')}(u) = \alpha_j^{(i'+1)}(u)$. Hence $\beta_j^{(w,i')}(u) = u(\alpha_j^{(w,i')}(u)) = u(\alpha_j^{(i'+1)}(u)) = \beta_j^{(i'+1)}(u)$. These equalities and (*) supra yield

$$\begin{aligned} \alpha_j^{(w,i)}(u) &= \theta_j^{(w,i)} \left(\beta_1^{(1)}(u), \dots, \beta_{\ell_0^{(w)}}^{(1)}(u), \dots, \beta_1^{(i)}(u), \dots, \beta_{\ell_{i-1}^{(w)}}^{(i)}(u) \right) \\ &= \theta_j^{(i)} \left(\beta_1^{(0)}(u), \dots, \beta_{\ell_0}^{(0)}(u), \dots, \beta_1^{(i)}(u), \dots, \beta_{\ell_i}^{(i)}(u) \right) = \alpha_j^{(i+1)}(u) \end{aligned}$$

Finally, for $b_1, \dots, b_\lambda \in B$, let $w = ((s_1, b_1), \dots, (s_k, b_k))$ and set $\omega(b_1, \dots, b_\lambda) = \omega^{(w)}(b_1, \dots, b_{\lambda(p_w)})$. \square

Finally, we can complete the wanted proof.

Proof of Theorem 3.7.

- *Case $k = 0$.* Then the function with empty graph \emptyset is in X_Φ and the empty constructor $\Theta = ()$ is such that $\pi_\Theta = \{\emptyset\}$ and satisfies the wanted conclusion.

For $k \geq 1$, we argue by induction on k .

- *Initial step of the induction: $k = 1$.* Apply Proposition 3.14.

- *Inductive step: going from $k-1$ to k (with $k \geq 2$).* If Φ is constant then the conclusion of the theorem is obviously true. Suppose Φ is not constant and let S be as in Proposition 3.15: S has at most $2k-1$ elements and the domain of every $u \in X_{\Phi,k}$ meets S . By Proposition 3.17, $\Phi^{(w)}$ is $(k-1)$ -bounded for every $w \in B^S$. The induction hypothesis yields a constructor $\Theta^{(w)}$ such that $\Phi^{(w)} = \Omega_{\Theta^{(w)}} \upharpoonright B^A$ and $p_w = \max(0, (k-1) - 1) = k-2$ (recall $k \geq 2$) and $\ell_i^{(w)} = 2((k-1) - i) - 1$ for $i = 0, \dots, p_w$. Now, Φ and $\Phi^{(w)}$ coincide on $[w]$ by definition of $\Phi^{(w)}$. Applying Proposition 3.18, we get a constructor Θ such that $\Phi = \Omega_\Theta \upharpoonright B^A$. Finally, $p = 1 + \max\{p_w \mid w \in B^S\} = k-1$ and $\ell_i = \max\{\ell_{i-1}^{(w)} \mid w \in B^S\} = 2(k-1 - (i-1)) - 1 = 2(k-i) - 1$. Lastly, $K = \sum_{i=0}^{i=k-1} \ell_i = \sum_{i=0}^{i=k-1} 2(k-i) - 1 = k^2$ and so π_Θ is a deterministic Φ -family which is k^2 -bounded. \square

4 Topology and bounded information

For the notion of uniformity on a space and the related classical results, we refer to Kelley's classical textbook [9].

4.1 The bounded uniformity on Baire like spaces

Proposition 4.1. *To any deterministic bounded covering π of B^A , associate*

$$\begin{aligned} U_\pi &= \{(f, g) \in B^A \times B^A \mid f, g \text{ extend the same function of } \pi\} \\ &= \bigcup_{u \in \pi} [u] \times [u] \quad (\text{cf. Notation 1.2}) \end{aligned}$$

1. *The family \mathcal{B} of all U_π 's is the basis of a transitive uniformity on B^A . We call it the bounded information uniformity.*
2. *The product topology (of the discrete topology on B) is associated to the bounded information uniformity.*

The bounded information uniformity is the right topological tool to characterize functionals using bounded information.

Theorem 4.2. *Endow B with the discrete uniformity and B^A with the bounded information uniformity. A total functional $\Phi : B^A \rightarrow B$ is uniformly continuous if and only if it uses uniformly bounded information.*

4.2 Curryfication and linear uniform continuity

Definition 4.3. *Endow B^A with the bounded information uniformity. We say that a total functional $\Psi : B^A \rightarrow B^A$ is linearly uniformly continuous if*

$$\begin{aligned} \exists k \quad \forall \ell \in \mathbb{N} \quad \forall \ell\text{-bounded covering } \pi \quad \exists k\ell\text{-bounded covering } \pi' \\ \forall f, g \in \mathbb{N}^{\mathbb{N}} \quad ((f, g) \in U_{\pi'} \Rightarrow (\Psi(f), \Psi(g)) \in U_\pi) \end{aligned}$$

Proposition 4.4. *Let $\Psi : B^A \rightarrow B^A$ be a total functional. Endow B with the discrete uniformity and B^A and $N^{A \cup \{\top\}}$ with the bounded information ones.*

1. *If Ψ is linearly uniformly continuous then it is uniformly continuous.*
2. *Ψ is linearly uniformly continuous if and only if $\partial\Psi : B^{A \cup \{\top\}} \rightarrow B$ is uniformly continuous (cf. Definition 1.3).*

5 Effective uniformly bounded information

The effectivization of uniform continuity is the expected one.

Definition 5.1. *A computable total functional $\Phi : B^A \rightarrow B$ is effectively uniformly continuous relative to the bounded uniformity on B^A if the inverse image of the diagonal of $B \times B$ contains a computable uniformly bounded covering of B^A (in other words, if there exists a computable (Φ, k) -family for some $k \in \mathbb{N}$).*

The proof of Theorem 3.7 uses a dichotomy: Φ is constant or not. This dichotomy is used recursively with the $\Phi^{(w)}$'s. Alas, to decide whether $\Phi^{(w)}$ is constant or not requires either the halting problem as oracle or the extra hypothesis that $X_{\Phi, k}$ is computable. The oracle leads to non computable functions $\theta_j^{(i)}$'s in the constructor Θ ! The extra hypothesis fails in general for ASM functionals, the ones we want to characterize! To get computable functions $\theta_j^{(i)}$'s, we give a different proof using Ramsey's theorem. The price is a super-exponential blow-up of k .

Theorem 5.2 (Ramsey's theorem). *Let $c \geq 2$. There exists a primitive recursive function $R : \mathbb{N}^c \rightarrow \mathbb{N}$ such that, for every coloring γ of the unordered pairs of $\{1, \dots, R(n_1, \dots, n_c)\}$ into c colors $1, \dots, c$, there exist a color i and a subset X_γ with n_i elements such that all pairs in X_γ have color i .*

The following Lemma is the key of the proof of Theorem 5.4.

Lemma 5.3. *Let $\Phi : B^A \rightarrow B$ be a total functional using k -bounded information. Let $(v_i)_{i=1, \dots, L}$, $(w_i)_{i=0, \dots, L}$ be sequences in $\mathfrak{F}_k^{A \rightarrow B}$ such that, for all i ,*

1. $v_i \in X_{\Phi, k}$ and $\text{Domain}(w_i) = \text{Domain}(v_i)$,
2. if $i < j$ then w_i and w_j agree on the common part of their domains,
3. w_0, \dots, w_L pairwise agree on the common part of their domains.

Suppose $L \geq R(n_0, \dots, n_k)$, the Ramsey number where $n_c = 1 + \binom{k}{c}(k+1)$ for $c = 0, \dots, k-1$ and $n_k = 2$. Then Φ is constant on $[\mathbf{w}]$ where $\mathbf{w} = w_0 \oplus \dots \oplus w_L$.

Proof. For $1 \leq i < j \leq L$ let $\gamma(i, j)$ be the number of elements in $\text{Domain}(v_i) \cap \text{Domain}(v_j)$. This function γ is a coloring of $\{1, \dots, L\}$ into $k+1$ colors $0, \dots, k$. Since the v_ℓ 's have $\leq k$ elements in their domains, γ is a coloring of pairs in $\{1, \dots, L\}$ into $k+1$ colors $0, \dots, k$. The hypothesis on L and Ramsey's theorem 5.2 insure the existence of a color c and a subset X of $\{1, \dots, L\}$ having n_c elements (cf. formula (*) supra) on which all pairs have color c . We argue on c .

• *Case $c = k$.* Since $n_k = 2$ there are $i < j$ such that $\text{Domain}(v_i) = \text{Domain}(v_j)$. Now, by definition, since $i < j$, v_j and w_i agree on the common part of their domains. Since $\text{Domain}(w_i) = \text{Domain}(v_i)$ we have $w_i = v_i$. Since $v_i \in X_\Phi$, we see that Φ is constant on $[w_i]$ hence is constant on $[\mathbf{w}]$.

• *Case $0 \leq c < k$.* The v_i 's, $i \in X$, are functions in Φ, k such that the domains of any pair of these functions have an intersection with exactly c points. Let ℓ be the least element of X . For each subset D of $\text{Domain}(v_\ell)$ with c elements, let $X_D = \{i \in X \setminus \{\ell\} \mid \text{Domain}(v_\ell) \cap \text{Domain}(v_i) = D\}$. The X_D 's form a partition of X . Since $n_c = 1 + \binom{k}{c}(k+1)$ there are $(k+1)\binom{k}{c}$ points in $X \setminus \{\ell\}$. Since there are $\binom{k}{c}$ possible subsets D , we see that there is a particular subset D such that X_D has at least $k+1$ elements. In particular, if $i, j \in X_D$, $i < j$, then $\text{Domain}(v_i) \cap \text{Domain}(v_j) = D$. Now, all v_i 's, $i \in X_D$ agree with w_ℓ hence $v_i \upharpoonright D = w_\ell \upharpoonright D$. Proposition 3.13 insures that Φ is constant on $w_\ell \upharpoonright D$. A fortiori, Φ is constant on $[\mathbf{w}]$.

Main Theorem 5.4. *There exists a primitive recursive total function $L : \mathbb{N} \rightarrow \mathbb{N}$ having the following property. Suppose Φ is a computable total functional $B^A \rightarrow B$ and π is a computable (Φ, k) -family for some $k \in \mathbb{N}$. Then there exist elements s_1, \dots, s_k in A , computable total functions $\theta_j^{(i)} : B^{ik} \rightarrow A$, with $i = 1, \dots, L(k)$, $j = 1, \dots, k$, and a computable total function $\omega : B^{(1+L(k))k} \rightarrow B$ such that $\Phi = \Omega_\Theta \upharpoonright B^A$ and π_Θ is a $(\Phi, k(1+L(k)))$ -family, where Θ is the constructor $\Theta = (s_1, \dots, s_k, \theta_1^{(1)}, \dots, \theta_k^{(1)}, \dots, \theta_1^{(L(k))}, \dots, \theta_k^{(L(k))}, \omega)$.*

2. *The above result is uniform in Φ, π : if Φ and π depend computably on a parameter $m \in \mathbb{N}$ then ω and the s_i 's, $\theta_j^{(i)}$'s are uniformly computable in m .*

Proof. Let $(u_i)_{i \in \mathbb{N}}$ be a computable enumeration of π . Let $L(k)$ be the Ramsey number L of Lemma 5.3. We fix some computable total ordering on A and define by induction on $i \geq 1$ the s_j 's and $\theta_j^{(i)}$'s, together with $\rho_i : B^{ik} \rightarrow \mathbb{N}$. Let $\mathbf{b} = (b_1, \dots, b_{k(L(k)+1)}) \in B^{k(L(k)+1)}$. Let us denote by $\mathbf{b}|ik$ the sequence $(b_1, \dots, b_{ik}) \in B^{ik}$. We define inductively the $\theta_j^{(i)}(\mathbf{b}|ik)$.

1. Choose s_1, \dots, s_k (non necessarily distinct) such that $\{s_1, \dots, s_k\} = \text{Domain}(u_0)$.
2. Let $\rho_1(\mathbf{b}|k)$ be the least integer ℓ such that u_ℓ and $(s_k, b_k) \oplus \dots \oplus (s_1, b_1)$ agree on the common part of their domains (recall Definition 1.1 and observe the inversion done in the sequence) and let $\theta_1^{(1)}(\mathbf{b}|k), \dots, \theta_k^{(1)}(\mathbf{b}|k)$ (non necessarily distinct) be such that $\{\theta_1^{(1)}(\mathbf{b}|k), \dots, \theta_k^{(1)}(\mathbf{b}|k)\} = \text{Domain}(u_{\rho_1(\mathbf{b}|k)})$.
3. Suppose the $\theta_j^{(i')}(\mathbf{b}|i'k)$'s, and the $\rho_{\ell_{\mathbf{b}|i'k}}$'s have been defined for $i' < i$. Let us write $s_{i'k+j} = \theta_j^{(i')}(\mathbf{b}|i'k)$. Let $\rho_i(\mathbf{b}|ki)$ be the least integer ℓ such that u_ℓ and $(s_{ik}, b_{ik}) \oplus \dots \oplus (s_1, b_1)$ agree on the common part of their domains and let $\theta_1^{(i)}(\mathbf{b}|ik), \dots, \theta_k^{(i)}(\mathbf{b}|ik)$ (non necessarily distinct) be such that $\{\theta_1^{(i)}(\mathbf{b}|ik), \dots, \theta_k^{(i)}(\mathbf{b}|ik)\} = \text{Domain}(u_{\rho_i(\mathbf{b}|ik)})$.

Let $v_i = u_{\rho_i(\mathbf{b}|ik)}$ and $w_i = ((s_{ik+k}, b_{ik+k}) \oplus \dots \oplus (s_1, b_1)) \upharpoonright \text{Domain}(v_i)$. Then all conditions of Lemma 5.3 are satisfied: $\text{Domain}(v_i) = \text{Domain}(w_i)$, the w_i 's pairwise agree (due to the inversion in the \oplus sum) and w_i agrees with v_j for $i < j$ by definition of $\rho_j(\mathbf{b}|jk)$. Thus, Φ is constant on $w_{kL(k)} \oplus \dots \oplus w_0$. Since $\text{Domain}(w_K)$ is the set of the $\theta_j^{(i)}(\mathbf{b}|ik)$'s, we see that Φ is constant on all $u \in \pi_\Theta$. Finally, we show that the values of Φ are given by some computable ω . Define ω as follows. Let $\zeta : A \rightarrow B$ be any constant function. For $\mathbf{b} \in B^{k(L(k)+1)}$, let $u = w_{kL(k)} \oplus \dots \oplus w_0$ be as above and let $f_{\mathbf{b}} = \zeta \oplus u$ and set $\omega(\mathbf{b}) = \Phi(f_{\mathbf{b}})$. By the above analysis, we know that Φ is constant on $[u]$. Since $f_{\mathbf{b}} \in [u]$, this constant value is given by $\omega(\mathbf{b})$. To conclude, recall that Φ is computable and observe that $f_{\mathbf{b}}$ is a computable function, uniformly in \mathbf{b} , hence θ is computable.

Corollaries 3.8 and 3.9 have effective versions.

Corollary 5.5. *Suppose $\Phi : B^A \rightarrow B$ is a computable total functional. The following conditions are equivalent.*

1. Φ is effectively uniformly continuous.
2. Φ admits a computable uniformly bounded Φ -family.
3. Φ admits a computable deterministic uniformly bounded Φ -family.
4. $\Phi = \Omega_\Theta \upharpoonright B^A$ for some computable constructor Θ as in Definition 3.4.

Corollary 5.6. *If a total functional $\Phi : B^A \rightarrow B$ is computable and effectively uniformly continuous then there exists a computable total functional $\Psi : B^A \rightarrow \mathcal{F}^{A \rightarrow B}$ such that,*

- (a) Ψ is computable and effectively uniformly continuous.
- (b) The range of Ψ is a computable deterministic uniformly bounded Ψ -family and Φ -family, i.e. for all $f \in B^A$, Ψ and Φ are constant on $[\Psi(f)]$.
- (c) For all $f \in B^A$, $f \in [\Psi(f)]$.

6 Characterization of ASM functionals

Proposition 6.1. *Let $T = \prod_{i=1, \dots, n} B_i^{A_i}$ be a product of function spaces where the A_i, B_i 's are countable. Let A, B be the disjoint sums of the A_i 's and the B_i 's respectively, so that B^A is a (proper) subset of T . Let $\Phi : T \rightarrow C$ a total functional which uses k -bounded information. Then there exists a total functional $\tilde{\Phi} : B^A \rightarrow C$ which also uses $(k + 1)$ -bounded information and is such that $\Phi = \tilde{\Phi} \upharpoonright T$. Moreover, if Φ is computable and admits a computable (Φ, k) -family then $\tilde{\Phi}$ is computable and admits a computable $(\tilde{\Phi}, k)$ -family.*

Main Theorem 6.2. *Let $T = \prod_{i=1, \dots, n} B_i^{A_i}$ be a product of function spaces where the A_i, B_i 's are countable. Let $\Psi : T \rightarrow T$ be a total functional. The following conditions are equivalent.*

1. Ψ uses uniformly bounded information.
2. There exists an ASM on the sorts occurring in T (with a suitable static framework) such that Ψ is exactly the functional associated to this ASM.

Moreover, if Ψ is computable and effectively linearly uniformly computable (i.e. there is some computable uniformly bounded $\partial\Psi$ -family) then the static framework of the ASM consists of computable functions.

Proof. Use Theorems 3.7, 4.2, 5.4 to $\tilde{\partial\Psi}$.

References

1. Andreas Blass and Yuri Gurevich. Abstract state machines capture parallel algorithms. *ACM Transactions on Computational Logic*, 4(4):578–651, 2003.
2. Andreas Blass and Yuri Gurevich. Abstract state machines capture parallel Algorithms: correction and extension. *ACM Transactions on Computational Logic*, 9(3), 32 p., 2008.
3. Nachum Dershowitz and Yuri Gurevich. A natural axiomatization of computability and proof of Church's Thesis. *Bulletin. of Symbolic Logic*, 14(3):299–350, 2008.
4. Marie Ferbus and Serge Grigorieff. ASM and operational algorithmic completeness of Lambda Calculus. *LNCS 6300*, 301–327, 2010.
5. Serge Grigorieff and Pierre Valarcher. Evolving MultiAlgebras unify all usual sequential computation models. *STACS 2010*, 2009.
6. Yuri Gurevich. Evolving Algebras: An Introductory Tutorial. *Bul. EATCS*, 43: 264–284, 1991.
7. Yuri Gurevich. The Sequential ASM Thesis. *Bul. EATCS*, 67: 93–124, 1999.
8. Yuri Gurevich. Sequential Abstract State Machines capture Sequential Algorithms. *ACM Transactions on Computational Logic*, 1(1):77–111, July 2000.
9. John L. Kelley. *General topology*. Springer-Verlag.

Topological Characterization of Gurevich Modelization of Sequential Algorithms

Version with full proofs

+ Appendix on uniformities
same numbering of sections and statements
than in the submitted 12 pages paper

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Abstract

Sequential algorithms over countable data structures are modeled by Gurevich's Abstract State Machines as the iteration of a functional which goes from a finite product of function spaces into itself. ASM functionals have the following properties for some k : (1) $_k$ they modify their argument on at most k points, (2) $_k$ their modulus of continuity is k -bounded. We show that these properties characterize ASM functionals and that property (2) $_{k^2}$ then holds with a modulus of continuity given by ground terms. This $k \mapsto k^2$ blow-up is optimal. We also give a topological interpretation of property (2) in terms of uniform continuity. The effective version also holds but is harder to prove and involves a super-exponential blow-up due to the use of Ramsey's theorem.

Keywords Specification; Abstract state machines; Logic in computer science; Computability; Theory of algorithms.

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1 Introduction

1.1 Roadmap

We consider deterministic sequential algorithms working with countable data structures. Here sequential means discrete time plus uniformly bounded action (as opposed to “wild” parallel action). This a priori quite informal notion of algorithm has been convincingly modeled around 1984 by Yuri Gurevich with Abstract State Machines (aka Evolving Algebras), see [8, 9, 10, 11, 12, 13]. A (deterministic) ASM consists of the following data.

- A multisort logical functional structure $\mathfrak{M} = (M_1, \dots, M_n; \mathbf{f}_1, \dots, \mathbf{f}_p)$ on a so-called static functional vocabulary (f_1, \dots, f_p) . This static vocabulary is typed: types are of the form M_j or $M_{j_1} \times \dots \times M_{j_q} \rightarrow M_r$. The interpretations \mathbf{f}_i 's respect the types of the f_i 's.
- A so-called dynamic functional vocabulary which is similarly typed.

- An ASM program consisting of assignments $f(t_1, \dots, t_\ell) := u$ where f is a dynamic symbol and the t_i 's and u are ground terms constructed with both vocabularies, conditionals and finite blocks of such instructions.

A state of an ASM is an interpretation of the dynamic vocabulary. A run of an ASM is a sequence of states obtained from the first one (the “initial” state) by iterated application of the ASM program. Thus,

- The base set of the structure and the interpretations of the static symbols are fixed. They represent the framework of the algorithm and the primitive operations “given for free”, let them be computable or not.
- The interpretations of the dynamic symbols vary from state to state. They represent the dynamic environment of the algorithm.

In this way, an algorithm appears as a static framework plus a functional mapping the interpretations of the dynamic symbols in some state into the interpretations in the next state (obtained by application of the ASM program). This functional is of the form $\Psi : T \rightarrow T$ with

$$T = \prod_{i \in I} \left(\left(\prod_{j \in J_i} M_j \right) \rightarrow M_{\ell_i} \right)$$

where I and the J_i 's are finite multisubsets of $\{1, \dots, n\}$ (recall n is the number of sorts of \mathfrak{M}), and the ℓ_i 's are in $\{1, \dots, n\}$.

Let k be the total number of occurrences of dynamic symbols in the terms of the ASM program. Governed by the ASM program, the ASM functional has the following properties:

- (1) _{k} It modifies its argument $\mathbf{f} = (f_1, \dots, f_q)$ on a set $X_{\mathbf{f}}$ of at most k points.
- (2) _{k} The points of $X_{\mathbf{f}}$ and the new values of \mathbf{f} on them depend only on the values of \mathbf{f} on a set $Y_{\mathbf{f}}$ of at most k points. In other words, the modulus of continuity is k -bounded.

In this work, we show (Theorems 3.7, 6.2) that any functional Ψ satisfying properties (1) _{k} and (2) _{k} is an ASM functional. More precisely, one can find some ASM static framework on the multisort domain (M_1, \dots, M_n) such that properties (1) _{k^2} and (2) _{k^2} hold in an ASM-like way: the k^2 points of $X_{\mathbf{f}}$ and the new values of \mathbf{f} on them are given by ground terms built with \mathbf{f} and the ASM static framework, cf. Theorem 3.7, Corollaries 3.8, 3.9. This $k \mapsto k^2$ blow-up cannot be improved (cf. Proposition 3.3).

This characterization can also be seen as a topological one using a particular

transitive uniformity compatible with the usual product topology on function spaces. Namely, ASM functionals $\Psi : T \rightarrow T$ are exactly those for which, for some k , there exist $2k$ uniformly continuous functionals $A_i, B_i : T \rightarrow M_j$ such that

$$\Psi(f)(x) = \begin{cases} B_i(f) & \text{if } x = A_i(f) \text{ for some } i = 1, \dots, k \\ f(x) & \text{otherwise} \end{cases}$$

A variant topological characterization uses a notion of “linear” uniform continuity of Ψ .

All these results hold in an effective version (Theorems 5.4, 6.2) with functionals associated to ASMs with computable static functions. However, the blow-up $k \mapsto k^2$ becomes a super-exponential blow-up since we have to use a completely different proof involving Ramsey’s theorem about k -colorings.

As a perspective work (in progress), these results seem to also apply to parallel algorithms, cf. Blass & Gurevich [1, 2].

1.2 Conventions and notations

Convention. The notation $f : A \rightarrow B$ is used for total functions from A to B whereas $u : A \rightarrow B$ is used for partial functions. In practice, we shall have $A = \mathbb{N}$ or $A = \mathbb{N} \cup \{\top\}$ and $B = \mathbb{N}$. We shall use letters Φ, Ψ as respective names for functionals $\Phi : B^A \rightarrow B$ and $\Psi : B^A \rightarrow B^A$.

Definition 1.1. 1. If $\xi, \eta : A \rightarrow B$ are partial functions, we let $\xi \oplus \eta : A \rightarrow B$ be the partial function such that $\text{Domain}(\xi \oplus \eta) = \text{Domain}(\xi) \cup \text{Domain}(\eta)$, $\xi \oplus \eta$ extends η and $(\xi \oplus \eta)(x)$ is equal to $\xi(x)$ if $x \in \text{Domain}(\xi) \setminus \text{Domain}(\eta)$.
2. We write (a, b) for the function with domain $\{a\}$ which maps a onto b and $((a_1, b_1), \dots, (a_n, b_n))$ in place of $(a_1, b_1) \oplus \dots \oplus (a_n, b_n)$.

Definition 1.2. 1. $\mathfrak{F}^{A \rightarrow B}$ (resp. $\mathfrak{F}_k^{A \rightarrow B}$) is the family of all partial functions $A \rightarrow B$ with finitely many (resp. at most k) points in their domains. If $u \in \mathfrak{F}^{A \rightarrow B}$, we let $[u]$ be the set of total functions $A \rightarrow B$ which extend u . If a_1, \dots, a_n are distinct, we let $((a_1, b_1), \dots, (a_n, b_n))$ be the partial function $u \in \mathfrak{F}_n^{A \rightarrow B}$ with domain $\{a_1, \dots, a_n\}$ such that $u(a_i) = b_i$ for $i = 1, \dots, n$.
2. We write $\mathfrak{F}, \mathfrak{F}_k$ when $A = B = \mathbb{N}$ and $\mathfrak{F}^\top, \mathfrak{F}_k^\top$ when $A = \mathbb{N} \cup \{\top\}, B = \mathbb{N}$.

Definition 1.3 (Curryfication). 1. If $\Psi : B^A \rightarrow B^A$ is a total functional, we denote by $\partial\Psi$ the total functional $\partial\Psi : (B^A \times A) \rightarrow B$ such that $(\partial\Psi)(f, a) = \Psi(f)(x)$ for all $f \in B^A$ and $a \in A$.
2. We identify A with $\{\top\} \rightarrow A$ and $B^A \times A$ with $B^{A \cup \{\top\}}$, and $\partial\Psi$ with a total functional $\partial\Psi : B^{A \cup \{\top\}} \rightarrow B$.

2 Continuity and finite information

First, we reformulate continuity of functionals in terms of how much information is used to compute their values. The core of the paper is to replace “finite” by “at most k ”.

Definition 2.1. *We use Definitions 1.2, 1.3.*

1. A covering π of B^A is a subfamily of $\mathfrak{F}^{A \rightarrow B}$ such that $B^A = \bigcup_{u \in \pi} [u]$. A covering π is deterministic if the $[u]$'s, $u \in \pi$, are pairwise disjoint, hence form a partition of B^A . 1. A covering π of B^A is a subfamily of $\mathfrak{F}^{A \rightarrow B}$ such that $B^A = \bigcup_{u \in \pi} [u]$. A covering π is deterministic if the $[u]$'s, $u \in \pi$, are pairwise disjoint, hence form a partition of B^A .

2. Let $\Phi : B^A \rightarrow B$ be a total functional. A [deterministic] Φ -family is a [deterministic] covering π such that Φ is constant on $[u]$ for every $u \in \pi$. We say that Φ uses [deterministic] finite information if there exists a [deterministic] Φ -family.

2. Let $\Psi : B^A \rightarrow B^A$ be a total functional. A [deterministic] Ψ -family is a [deterministic] $\partial\Psi$ -family. We say that Ψ uses [deterministic] finite information if so does $\partial\Psi$.

Proposition 2.2. *A total functional $\Psi : B^A \rightarrow B^A$ uses finite information if and only if so does, for each $a \in A$, the $B^A \rightarrow B$ functional $f \mapsto \Psi(f)(a)$.*

Using finite information is a mere reformulation of continuity.

Proposition 2.3. *Let A be \mathbb{N} or $\mathbb{N} \cup \{\top\}$ and B be a countable set with at least two elements (in practice \mathbb{N} or $\{0, 1\}$). On B we consider the discrete topology and on B^A we consider the usual product topology for which the family of sets $[u]$, $u \in \mathfrak{F}^{A \rightarrow \mathbb{N}}$, constitutes a basis of open sets. For a total functional $B^A \rightarrow B$ or $B^A \rightarrow B^A$, the following conditions are equivalent.*

- (i) *to be continuous.*
- (ii) *to use finite information.*
- (iii) *to use deterministic finite information.*

Proof. We treat the case $\Phi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$. Implication (iii) \Rightarrow (ii) is obvious.

(i) \Leftrightarrow (ii). Φ is continuous if and only if, for every $f \in \mathbb{N}^{\mathbb{N}}$, there exists an open set hence also some $u \in \mathfrak{F}$ such that $f \in [u]$ and Φ is constant on $[u]$.

(ii) \Rightarrow (iii). In Definition 2.1 one can require that the domain of each $u \in \pi$ is an initial finite segment of \mathbb{N} : if not, replace u by all its possible extensions to $\{0, \dots, \max(\text{Domain}(u))\}$. A deterministic Φ -family is the family of functions $u \in \mathfrak{F}$ with domain an initial segment of \mathbb{N} such that Φ is constant on $[u]$ and which are minimal with respect to function extension. \square

3 Uniformly bounded information

3.1 From finite to uniformly bounded information

Definition 3.1. Let $\Phi : B^A \rightarrow B$ be a total functional.

1. Let $k \in \mathbb{N}$. A [deterministic] k -covering π of B^A is a [deterministic] covering (cf. Definition 2.1) included in $\mathfrak{F}_k^{A \rightarrow B}$. A covering is uniformly bounded if it is a k -covering for some $k \in \mathbb{N}$.

2. A (Φ, k) -family [(Φ , bounded)-family] is a Φ -family which is a k -covering [uniformly bounded covering]. We say that Φ uses [deterministic] k -bounded information if there exists a [deterministic] (Φ, k) -family. Φ uses [deterministic] uniformly bounded information if it uses [deterministic] uniformly bounded information.

Example 3.2. A trivial example of a deterministic k -covering of $\mathbb{N}^{\mathbb{N}}$ is that of all functions having some fixed domain $\{s_1, \dots, s_k\}$ with k elements. A less trivial example, with $k = 2$, is the family of all functions u with graph of the form $\{(a, y), (\theta(y), z)\}$ (in other words, $\text{Domain}(u) = \{a, \theta(u(a))\}$) where $a \in \mathbb{N}$ and $\theta : \mathbb{N} \rightarrow \mathbb{N}$ are fixed but y, z vary in \mathbb{N} .

The analog of equivalence (ii) \Leftrightarrow (iii) of Proposition 2.3 fails with respect to bounded information. The next result shows that, in general, the existence of a (Φ, k) -family does not imply that of a deterministic (Φ, k) -family. A quadratic blow-up is unavoidable (it is also sufficient, cf. Theorem 3.7).

Proposition 3.3. *There exists a total functional $\Phi : \mathbb{N}^{\mathbb{N}} \rightarrow \{0, 1\}$ which uses k -bounded information and deterministic k^2 -bounded information but cannot use only deterministic $(k^2 - 1)$ -bounded information.*

Proof. We write $\mathbf{n} = \{0, \dots, n - 1\}$. Let

$$\Phi(f) = \begin{cases} 0 & \text{if } f(ik) = f(ik + 1) = \dots = f(ik + (k - 1)) = 0 \\ & \text{for some } i \in \{0, \dots, k - 1\} \\ 1 & \text{otherwise} \end{cases}$$

To see that Φ uses k -bounded information, observe that

$$\begin{aligned} \Phi^{-1}(0) &= \bigcup_{0 \leq i < k} [(ik, 0), (ik + 1, 0), \dots, (ik + (k - 1), 0)] \\ \Phi^{-1}(1) &= \bigcup_{\substack{0 \leq j_0, \dots, j_{k-1} < k \\ a_0, \dots, a_{k-1} \neq 0}} [(j_0, a_0), (k + j_1, a_1), \dots, ((k - 1)k + j_{k-1}, a_{k-1})] \end{aligned}$$

Since $\Phi(f)$ depends only on the values of f on \mathbf{k}^2 , there is an obvious deterministic (Φ, k^2) -family, namely $\mathbb{N}^{\mathbf{k}^2}$ (i.e. all functions with domain \mathbf{k}^2).

We now show that there is no deterministic $(\Phi, k^2 - 1)$ -family π . Suppose, by way of contradiction, that there is such a π . The idea of the proof is to go from π to a “finitary” family π'' of partial functions $\mathbf{k}^2 \rightarrow \{0, 1\}$ having domains with exactly $k^2 - 1$ points and then use a counting argument to get a contradiction. For a partial function $u : \mathbf{k}^2 \rightarrow \{0, 1\}$, let $\widetilde{[u]}$ be the set of total functions $\mathbf{k}^2 \rightarrow \{0, 1\}$ which extend u . For $u \in \pi$, let u_{k^2} be the restriction of u to $\text{Domain}(u) \cap \mathbf{k}^2$. We face a problem: $[u] \cap [v] = \emptyset$ does not imply $\widetilde{[u_{k^2}]} \cap \widetilde{[v_{k^2}]} = \emptyset$. This is why we consider

$$A = \{f \in \{0, 1\}^{\mathbb{N}} \mid f(x) = 0 \text{ for all } x \geq k^2\}, \quad \pi_A = \{u \in \pi \mid [u] \text{ meets } A\}.$$

Clearly, $A = \bigcup_{u \in \pi_A} [u] \cap A$ and every $u \in \pi_A$ takes values in $\{0, 1\}$ and takes value 0 on $\{x \in \text{Domain}(u) \mid x \geq k^2\}$. In particular, the $\widetilde{[u_{k^2}]}$'s, $u \in \pi_A$, are pairwise disjoint and their union is $\{0, 1\}^{\mathbf{k}^2}$. Also, the domain of each u_{k^2} has at most $k^2 - 1$ points. Lastly, since Φ is constant on $[u]$, for $u \in \pi$, and $\Phi(f)$ does depend on the sole values of f on \mathbf{k}^2 , Φ is also constant on $[u_{k^2}]$. For $\varepsilon = 0, 1$, let $F_\varepsilon = \{f \upharpoonright \mathbf{k}^2 \mid f \in \Phi^{-1}(\varepsilon)\}$. Then F_ε is a union of some $\widetilde{[u_{k^2}]}$'s, $u \in \pi_A$. Thus, $\pi' = \{u_{k^2} \mid u \in \pi_A\}$ is a family of partial functions $v : \mathbf{k}^2 \rightarrow \{0, 1\}$ defined on at most $k^2 - 1$ points such that F_0 and F_1 are disjoint unions of some $\widetilde{[v]}$'s. For each $v \in \pi'$, choose a set S with $k^2 - 1$ points such that $\text{Domain}(v) \subseteq S \subset \mathbf{k}^2$ and replace v by all its extensions to S . In this way, we get a family π'' such that F_0 and F_1 are disjoint unions of some $\widetilde{[w]}$'s with $w \in \pi''$. To conclude, let us do some counting. A total function $\alpha : \mathbf{k}^2 \rightarrow \{0, 1\}$ is in F_1 if and only if, for every $i = 0, \dots, k - 1$, $\alpha \upharpoonright \{ik, ik + 1, \dots, ik + (k - 1)\}$ is not the constant function with value 0. Thus, the cardinality of F_1 is $(2^k - 1)^k$ and that of F_2 is $2^{k^2} - (2^k - 1)^k$. Observe that these numbers are odd. However, each $\widetilde{[v]}$, for $v \in \pi''$, contains exactly 2 elements because there is only one point in $\mathbf{k}^2 \setminus \text{Domain}(v)$. Thus, as a disjoint union of some $\widetilde{[v]}$'s with $v \in \pi''$, both F_0, F_1 contain an even number of elements. Contradiction! \square

3.2 Constructors

The next Definition and Lemma extend the idea of Example 3.2 to get simply definable deterministic bounded coverings of B^A .

Definition 3.4. 1, A constructor is a tuple

$$\Theta = (s_1, \dots, s_{\ell_0}, \theta_1^{(1)}, \dots, \theta_{\ell_1}^{(1)}, \dots, \theta_1^{(p)}, \dots, \theta_{\ell_p}^{(p)}, \omega)$$

where, letting $\lambda(i) = \sum_{m=0}^{i-1} \ell_m$ for $i = 1, \dots, p$, and $\lambda = \sum_{i=1, \dots, p} \lambda(i)$,

- (i) $p, \ell_0, \ell_1, \dots, \ell_p \in \mathbb{N}$ and $\ell_i = 0 \Rightarrow \ell_{i+1} = 0$ for $i = 0, \dots, p-1$,
- (ii) the s_j 's are elements of A (the seeds of the constructor),
- (iii) the $\theta_j^{(i)}$'s are total functions $\theta_j^{(i)} : B^{\lambda^{(i)}} \rightarrow A$.
- (iv) ω is a total function $\omega : B^\lambda \rightarrow B$.

2. To Θ we associate partial functionals, for $0 \leq i \leq p$ and $1 \leq j \leq \ell_i$,

$$\alpha_j^{(i)} : (A \multimap B) \multimap A \quad \beta_j^{(i)} : (A \multimap B) \multimap B \quad \Omega_\Theta : (A \multimap B) \multimap B$$

inductively defined as follows: for $j \in \{1, \dots, \ell_i\}$, $\xi \in \mathfrak{F}^{A \multimap B}$,

$$\begin{aligned} \alpha_j^{(0)}(\xi) &= s_j & \beta_j^{(i)}(\xi) &= \xi(\alpha_j^{(0)}(\xi)) \quad \text{for } i = 0, \dots, p \\ \alpha_j^{(i)}(\xi) &= \theta_j^{(i)} \left((\beta_{j'}^{(i')}(\xi))_{\substack{i'=0, \dots, i-1 \\ j'=1, \dots, \ell_{i'}}} \right) & & \text{for } i = 1, \dots, p \\ \Omega_\Theta(\xi) &= \omega \left((\beta_j^{(i)}(\xi))_{\substack{i=0, \dots, p \\ j=1, \dots, \ell_i}} \right) \end{aligned}$$

3. $u \in \mathfrak{F}^{A \multimap B}$ is Θ -good if Ω_Θ is defined, i.e. if all $\alpha_j^{(i)}(u)$'s are defined and $\text{Domain}(u)$ contains $\mathfrak{s}_\Theta(u) = \bigcup_{i=0, \dots, p} \{\alpha_j^{(i)}(\xi) \mid 1 \leq j \leq \ell_i\}$.

The following result is straightforward.

Proposition 3.5. *Let Θ be a constructor as in Definition 3.4. A partial function u is minimal among Θ -good functions relative to function extension if and only if $\text{Domain}(u) = \mathfrak{s}_\Theta(u)$.*

Lemma 3.6. *The family π_Θ of minimal Θ -good functions is a deterministic $(\sum_{m=0}^{m=p} \ell_m)$ -covering.*

Proof. Clearly, the domain of every $u \in \pi_\Theta$ has at most $\sum_{m=0}^{m=p} \ell_m$ elements. To see that π_Θ is a covering, observe that every $f \in B^A$ extends $\mathfrak{s}(f)$. To see that π_Θ is deterministic, we show that any two distinct Θ -good functions u, v are necessarily incompatible. In case there exists (i, j) such that $\alpha_j^{(i)}(u) \neq \alpha_j^{(i)}(v)$, consider the minimum such i . Since $\alpha_j^{(0)}(u) = \alpha_j^{(0)}(v) = s_j$, we have $i \geq 1$. Now, $\alpha_j^{(i)}(u)$ is obtained as $\theta_j^{(i)}$ applied to the $\alpha_{j'}^{(i')}(u)$'s and the $\beta_{j'}^{(i')}(u)$'s with $0 \leq i' < i$ and $1 \leq j' \leq \ell_{i'}$. Idem with $\alpha_j^{(i)}(v)$. Inequality $\alpha_j^{(i)}(u) \neq \alpha_j^{(i)}(v)$ insures an inequality between some corresponding arguments in $\theta_j^{(i)}$ when computing with u and v . Since i is minimum, for $i' < i$ and $1 \leq j' \leq \ell_{i'}$, we have $\alpha_{j'}^{(i')}(u) = \alpha_{j'}^{(i')}(v)$ so that the corresponding arguments have to be $\beta_{j'}^{(i')}(u)$ and $\beta_{j'}^{(i')}(v)$ for some i', j' . Now, $\alpha_{j'}^{(i')}(u) = \alpha_{j'}^{(i')}(v)$ and $\beta_{j'}^{(i')}(u) = u(\alpha_{j'}^{(i')}(u))$ and $\beta_{j'}^{(i')}(v) = v(\alpha_{j'}^{(i')}(v))$, so that

inequality $\beta_{j'}^{(i')}(u) \neq \beta_{j'}^{(i')}(v)$ proves that u and v are incompatible. Finally, in case $\alpha_j^{(i)}(u) = \alpha_j^{(i)}(v)$ for all i, j 's then u, v have the same domain and the condition $u \neq v$ requires that they are incompatible. \square

3.3 Main theorem on uniformly bounded information

We can now prove that Proposition 3.3 is optimal: a quadratic blow-up is sufficient for determinization. This is the first main theorem of the paper.

Main Theorem 3.7. *If the total functional $\Phi : B^A \rightarrow B$ uses k -bounded information then it uses deterministic k^2 -bounded information and there exists a constructor Θ as in Definition 3.4 such that $\Phi = \Omega_\Theta \upharpoonright B^A$ and π_Θ is a deterministic (Φ, k^2) -family and*

- (a) *if $k = 0$ then Θ is reduced to (ω) and π_Θ is the singleton family consisting of the function with empty domain,*
- (b) *if $k \geq 1$ then $p = k - 1$ and $\ell_i = 2(k - i) - 1$ for $i = 0, \dots, p$.*

As immediate corollaries, we get.

Main Corollary 3.8. *Suppose $\Phi : B^A \rightarrow B$ is a total functional. The following conditions are equivalent.*

- (1) *Φ uses uniformly bounded information.*
- (2) *Φ uses deterministic uniformly bounded information.*
- (3) *There exists a constructor Θ as in Definition 3.4 such that $\Phi = \Omega_\Theta \upharpoonright B^A$.*

Proof. The sole non trivial implication is (1) \Rightarrow (3) which is Theorem 3.7. \square

Main Corollary 3.9. *If a total functional $\Phi : B^A \rightarrow B$ uses k -bounded information then there exists a total functional $\Psi : B^A \rightarrow \mathcal{F}_{k^2}^{A \rightarrow B}$ such that,*

- (a) *For all $f \in B^A$, Ψ and Φ are constant on $[\Psi(f)]$ and $f \in [\Psi(f)]$. Thus, the range of Ψ is a deterministic uniformly bounded Ψ -family and Φ -family.*
- (b) *Ψ is uniformly continuous.*

Proof. Let Θ be given by Theorem 3.7. Set

$$\Psi(f) = f \upharpoonright \mathfrak{s}_\Theta(f) = f \upharpoonright \bigcup_{i=0, \dots, p} \{\alpha_j^{(i)}(f) \mid 1 \leq j \leq \ell_i\}$$

where the $\alpha_j^{(i)}$'s are constructed as in Definition 3.4. Then the family of $\Psi(f)$'s, $f \in \mathbb{N}^{\mathbb{N}}$, is exactly π_{Θ} , which proves (a), (c) by Theorem 3.7). Finally, the computation of $\Psi(f)$ does not use the value of f on $\alpha_1^{(p)}$ (recall $\ell_p = 1$), it only uses the values of f on the $\alpha_j^{(i)}$'s for $i = 0, \dots, p-1$. Thus, Ψ uses only deterministic $(k^2 - 1)$ -bounded information. \square

3.4 Proof of Theorem 3.7

It is easy to see that the proof given for Proposition 3.3 does not preserve boundedness, a different argument is required to prove Theorem 3.7. We shall argue by induction on $k \geq 1$, the case $k = 0$ being special and trivial.

First, we introduce convenient notation and prove some auxiliary results.

Definition 3.10. *Suppose $\Phi : B^A \rightarrow B$ is a total functional. We let X_{Φ} [$X_{\Phi,k}$] be the family of partial functions $u : \mathbb{N} \rightarrow \mathbb{N}$ such that Φ is constant on $[u]$ and $\text{Domain}(u)$ is finite [has at most k elements]. We let $\varphi_{\Phi} : X_{\Phi} \rightarrow \mathbb{N}$ be such that, for $u \in X_{\Phi}$, $\varphi_{\Phi}(u)$ is the common value of all $\Phi(f)$'s for $f \in [u]$.*

Proposition 3.11. *If Φ uses k -bounded information ($k \geq 1$) then $X_{\Phi,k}$ is infinite.*

Proof. If $X_{\Phi,k}$ contains the empty domain function, then $X_{\Phi,k} = \mathfrak{F}_k^{A \rightarrow B}$. Else the restrictions in $X_{\Phi,k}$ of the constant functions in B^A are pairwise distinct. \square

Proposition 3.12. *If $u, v \in X_{\Phi}$ agree on $\text{Domain}(u) \cap \text{Domain}(v)$ (in particular, if v extends u or if they have disjoint domains) then $\varphi_{\Phi}(u) = \varphi_{\Phi}(v)$.*

Proof. Consider $f \in \mathbb{N}^{\mathbb{N}}$ which extends both u and v . Then $\Phi(f) = \varphi_{\Phi}(u)$ and $\Phi(f) = \varphi_{\Phi}(v)$ hence $\varphi_{\Phi}(u) = \varphi_{\Phi}(v)$. \square

The following result will be used for effectivization to prove Lemma 5.3.

Proposition 3.13. *Suppose Φ uses k -bounded information, u_1, \dots, u_{k+1} in $X_{\Phi,k}$, $D \subset A$, $v \in B^D$ are such that $D = \text{Domain}(u_i) \cap \text{Domain}(u_j)$ for all $i < j$ and $u_i \upharpoonright D = v$ for all i 's. Then Φ is constant on $[v]$.*

Proof. Applying Proposition 3.12, we see that $\varphi_{\Phi}(u_1) = \dots = \varphi_{\Phi}(u_{k+1})$. Let $f \in [v]$. Then $f \in [u]$ for some $u \in X_{\Phi,k}$. Since $\text{Domain}(u)$ has at most k elements there exists i such that $\text{Domain}(u) \cap \text{Domain}(u_i) \subseteq D$. Since u, u_i agree with v on D , Proposition 3.12 insures $\varphi_{\Phi}(u) = \varphi_{\Phi}(u_i) = \varphi_{\Phi}(u_1)$. Thus, $\Phi(f) = \varphi_{\Phi}(u_1)$, so Φ is constant on $[v]$. \square

The next Proposition gives the case $k = 1$ of Theorem 3.7.

Proposition 3.14. *If $\Phi : B^A \rightarrow B$ is a total functional using 1-bounded information then for every $s \in \bigcup_{u \in X_{\Phi,1}} \text{Domain}(u)$, the functional Φ is constant on $[u]$ for every u with domain $\{s\}$ (in other words, letting $\Theta = (s)$, π_{Θ} is a $(\Phi, 1)$ -family).*

Proof. *Case 1: the function \emptyset with empty domain is in $X_{\Phi,1}$.* Then Φ is constant. Though an empty Θ is OK, we can also set $\Theta = (s)$ for any s .

Case 2: Φ is not constant. Then all functions $u \in X_{\Phi,1}$ have a one element domain. By Proposition 3.12 with $k = 1$, no two of these domains can be disjoint. Thus, all u 's in $X_{\Phi,1}$ have the same domain $\{s\}$ for some s . \square

Proposition 3.15. *Let $\Phi : B^A \rightarrow B$ be a k -bounded total functional. If Φ is not constant then there exists a set S with at most $2k - 1$ elements such that the domain of every partial function $u \in X_{\Phi}$ meets S .*

Proof. Since Φ is not constant so is φ_{Φ} . Let $u, v \in X_{\Phi,k}$ be such that $\varphi_{\Phi}(u) \neq \varphi_{\Phi}(v)$. Set $S = \text{Domain}(u) \cup \text{Domain}(v)$. Applying Proposition 3.12, we see that $\text{Domain}(u)$ and $\text{Domain}(v)$ are not disjoint so that S has at most $2k - 1$ elements. Also, if $w \in X_{\Phi}$ then $\varphi_{\Phi}(w)$ is different from at least one of $\varphi_{\Phi}(u)$ and $\varphi_{\Phi}(v)$ hence $\text{Domain}(w)$ meets S . \square

Definition 3.16. *Suppose $\Phi : B^A \rightarrow B$ is a total functional and $S \subset A$ is finite and $w \in B^S$. We let $\Phi^{(w)} : B^A \rightarrow B$ be the total functional such that $\Phi^{(w)}(f) = \Phi(f \oplus w)$ for every $f \in B^A$.*

Proposition 3.17. *Let $\Phi : B^A \rightarrow B$ be a total functional using k -bounded information. Suppose that there is a non empty set S such that the domain of every function $u \in X_{\Phi,k}$ meets S . Then, for every function $w \in \mathbb{N}^S$, the total functional $\Phi^{(w)} : B^A \rightarrow B$ is $(k - 1)$ -bounded.*

Proof. For $f \in B^A$, apply the hypothesis on Φ to $f \oplus w$ (cf. Notation 1.1) to get some $u \in X_{\Phi,k}$ such that $f \oplus w$ extends u . Let \tilde{u} be the restriction of u to $\text{Domain}(u) \setminus S$. Observe that f extends \tilde{u} . Thus, to prove that $\Phi^{(w)}$ uses $(k - 1)$ -bounded information, it suffices to show that $\tilde{u} \in X_{\Phi^{(w)},k-1}$. The hypothesis on Φ and S insures that $\text{Domain}(u)$ meets S . Since $\text{Domain}(u)$ has at most k elements, there are at most $k - 1$ elements in $\text{dom}(\tilde{u})$. It remains to show that $\Phi^{(w)}$ is constant on $[\tilde{u}]$. Let $f, g \in [\tilde{u}]$. Since w is compatible with u hence with \tilde{u} , both $f \oplus w$ and $g \oplus w$ extend \tilde{u} and w hence both extend u . Since $u \in X_{\Phi,k}$, we have $\Phi(f \oplus w) = \Phi(g \oplus w)$ hence $\Phi^{(w)}(f) = \Phi^{(w)}(g)$. \square

Proposition 3.18. *Let S be a finite non empty subset of A and , for every $w \in B^S$, let $\Phi^{(w)} : B^A \rightarrow B$ be a total functional and*

$$\Theta^{(w)} = (s_1^{(w)}, \dots, s_{\ell_0^{(w)}}^{(w)}, \theta_1^{(w,1)}, \dots, \theta_{\ell_1^{(w)}}^{(w,1)}, \dots, \theta_1^{(w,p_w)}, \dots, \theta_{\ell_{p_w}^{(w)}}^{(w,p_w)}, \omega^{(w)})$$

be a constructor as in Definition 3.4 such that $\Phi^{(w)}$ is constant on $[u]$ for every Θ_w - good partial function u . Let $\Phi : B^A \rightarrow B$ be the unique total functional such that, for every $f \in B^A$, $\Phi(f) = \Phi^{(f|S)}(f)$. Suppose that there exists some $K \in \mathbb{N}$ which majorizes all $p^{(w)}$'s and $\ell_i^{(w)}$'s, for $w \in B^S$ and $0 \leq i \leq \ell_{p^{(w)}}$. Then there exists a constructor

$$\Theta = (s_1, \dots, s_{\ell_0}, \theta_1^{(1)}, \dots, \theta_{\ell_1}^{(1)}, \dots, \theta_1^{(p)}, \dots, \theta_{\ell_p}^{(p)}, \omega)$$

as in Definition 3.4 such that $\Phi = \Omega_{\Theta} \upharpoonright B^A$ and

- (i) $\{s_1, \dots, s_{\ell_0}\} = S$,
- (ii) $p = 1 + \max\{p_w \mid w \in B^S\}$, $\ell_i = \max\{\ell_{i-1}^{(w)} \mid p_w \geq i-1\}$ for $i = 1, \dots, p$.

Proof. Let p , the s_j 's and ℓ_i 's be as given by clause (ii). First, we define the total functions $\theta_j^{(i)}$'s, $i = 1, \dots, p$. Given $\sum_{i=0}^{i=p-1} \ell_i$ elements $b_\ell^{(i)}$ of \mathbb{N} , for $i = 1, \dots, p$, we get a tuple which can be an input for the $\theta_j^{(i)}$'s:

$$D^{(i)} = (b_1^{(0)}, \dots, b_{\ell_0}^{(0)}, \dots, b_1^{(i-1)}, \dots, b_{\ell_{i-1}}^{(i-1)})$$

For $w \in B^S$ and $1 \leq i \leq p_w$, we consider the tuple obtained from $D^{(i)}$ by removing the first blocks of b 's and trimming blocks of length ℓ_i to length $\ell_{i-1}^{(w)}$, so as to get a tuple which can be an input for the $\theta_j^{(w, i-1)}$'s:

$$E^{(w, i-1)} = (b_1^{(1)}, \dots, b_{\ell_0}^{(1)}, \dots, b_1^{(i-1)}, \dots, b_{\ell_{i-1}^{(w)}}^{(i-1)})$$

We are now ready to define the $\theta_j^{(i)}$'s.

Let $w \in B^S$ be such that $w(s_1) = b_1^{(0)}, \dots, w(s_{\ell_0}) = b_{\ell_0}^{(0)}$. Then, for $i = 1, \dots, p$ and $1 \leq j \leq \ell_i$, we set

$$(*) \quad \theta_j^{(i)}(D^{(i)}) = \begin{cases} s_j^{(w)} & \text{if } i = 1, j \leq \ell_0^{(w)} \\ \theta_j^{(w, i-1)}(E^{(w, i-1)}) & \text{if } 1 \leq i-1 \leq p_w, j \leq \ell_{i-1}^{(w)} \\ b_1^{(0)} & \text{if } i-1 > p_w \text{ or } j > \ell_{i-1}^{(w)} \end{cases}.$$

Now that the $\theta_j^{(i)}$'s are defined, let us prove that Φ is constant on $[u]$ for every $u \in \pi_{\Theta}$. Let $f, g \in [u]$, we have to prove that $\Phi(f) = \Phi(g)$. Since the seeds of Θ are $S = \{s_1, \dots, s_{\ell_0}\}$, the domain of u contains S . Let w be the restriction of u to S . The hypothesis of the Proposition insures that Φ and $\Phi^{(w)}$ coincide on $[w]$. In particular, they coincide on $[u]$ hence $\Phi(f) = \Phi^{(w)}(f)$ and $\Phi(g) = \Phi^{(w)}(g)$. Since $u \in \pi_{\Theta}$, the $\alpha_j^{(i)}(u)$'s for $i = 0, \dots, p$ and $1 \leq j \leq \ell_i$, are all defined and belong to $\text{Domain}(u)$. Now, arguing with $\Phi^{(w)}$ and the

$\alpha_j^{(w,i)}$'s, we show, by induction on $i = 0, \dots, p_w$ that $\alpha_j^{(w,i)}(u) = \alpha_j^{(i+1)}(u)$ for all $j = 1, \dots, \ell_i^{(w)}$.

Initial case $i = 0$. Using the definitions of $\alpha_j^{(w,0)}(u)$ and $\theta_1^{(w)}$, we get

$$\begin{aligned} \alpha_j^{(w,0)}(u) &= s_j^{(w)} = \theta_j^{(1)}(D^{(1)}) = \theta_j^{(1)}(b_1^{(0)}, \dots, b_{\ell_0}^{(0)}) \\ &= \theta_j^{(1)}(w(s_1), \dots, w(s_{\ell_0})) \\ &= \alpha_j^{(1)}(w) \\ &= \alpha_j^{(1)}(u) \quad \text{since } u \text{ extends } w. \end{aligned}$$

Inductive step: from $0 \leq i-1 < p_w$ to i . We have

$$\alpha_j^{(w,i)}(u) = \theta_j^{(w,i)} \left(\beta_1^{(w,0)}(u), \dots, \beta_{\ell_0^{(w)}}^{(w,0)}(u), \dots, \beta_1^{(w,i-1)}(u), \dots, \beta_{\ell_{i-1}^{(w)}}^{(w,i-1)}(u) \right)$$

By induction hypothesis, for $i' < i$, $\alpha_j^{(w,i')}(u) = \alpha_j^{(i'+1)}(u)$. Hence $\beta_j^{(w,i')}(u) = u(\alpha_j^{(w,i')}(u)) = u(\alpha_j^{(i'+1)}(u)) = \beta_j^{(i'+1)}(u)$. These equalities and (*) supra yield

$$\begin{aligned} \alpha_j^{(w,i)}(u) &= \theta_j^{(w,i)} \left(\beta_1^{(1)}(u), \dots, \beta_{\ell_0^{(w)}}^{(1)}(u), \dots, \beta_1^{(i)}(u), \dots, \beta_{\ell_{i-1}^{(w)}}^{(i)}(u) \right) \\ &= \theta_j^{(i)} \left(\beta_1^{(0)}(u), \dots, \beta_{\ell_0}^{(0)}(u), \dots, \beta_1^{(i)}(u), \dots, \beta_{\ell_i}^{(i)}(u) \right) \\ &= \alpha_j^{(i+1)}(u) \end{aligned}$$

Finally, for $b_1, \dots, b_\lambda \in B$, let $w = ((s_1, b_1), \dots, (s_k, b_k))$ and set $\omega(b_1, \dots, b_\lambda) = \omega^{(w)}(b_1, \dots, b_{\lambda(p_w)})$. \square

Finally, we can complete the wanted proof.

Proof of Theorem 3.7.

- *Case $k = 0$.* Then the function with empty graph \emptyset is in X_Φ and the empty constructor $\Theta = ()$ is such that $\pi_\Theta = \{\emptyset\}$ and satisfies the wanted conclusion. For $k \geq 1$, we argue by induction on k .
- *Initial step of the induction: $k = 1$.* Apply Proposition 3.14.
- *Inductive step: going from $k-1$ to k (with $k \geq 2$).* If Φ is constant then the conclusion of the theorem is obviously true. Suppose Φ is not constant and let S be as in Proposition 3.15: S has at most $2k-1$ elements and the domain of every $u \in X_{\Phi,k}$ meets S . By Proposition 3.17, $\Phi^{(w)}$ is $(k-1)$ -bounded for every $w \in B^S$. The induction hypothesis yields a constructor $\Theta^{(w)}$ such that $\Phi^{(w)} = \Omega_{\Theta^{(w)}} \upharpoonright B^A$ and $p_w = \max(0, (k-1) - 1) = k-2$ (recall $k \geq 2$) and $\ell_i^{(w)} = 2((k-1) - i) - 1$ for $i = 0, \dots, p_w$. Now, Φ and $\Phi^{(w)}$ coincide on $[w]$ by definition of $\Phi^{(w)}$. Applying Proposition 3.18, we get a constructor Θ such that $\Phi = \Omega_\Theta \upharpoonright B^A$. Finally, $p = 1 + \max\{p_w \mid w \in B^S\} = k-1$ and

$\ell_i = \max\{\ell_{i-1}^{(w)} \mid w \in B^S\} = 2(k-1-(i-1)) - 1 = 2(k-i) - 1$. Lastly, $K = \sum_{i=0}^{i=k-1} \ell_i = \sum_{i=0}^{i=k-1} 2(k-i) - 1 = k^2$ and so π_Θ is a deterministic Φ -family which is k^2 -bounded. \square

4 Topology and bounded information

For the notion of uniformity on a space and the related classical results, we refer to classical textbooks (Bourbaki or Kelley's [3, 14]) and the Appendix.

4.1 The bounded uniformity on Baire like spaces

Proposition 4.1. *To any deterministic bounded covering π of B^A , associate*

$$\begin{aligned} U_\pi &= \{(f, g) \in B^A \times B^A \mid f, g \text{ extend the same function of } \pi\} \\ &= \bigcup_{u \in \pi} [u] \times [u] \quad (\text{cf. Notation 1.2}) \end{aligned}$$

1. *The family \mathcal{B} of all U_π 's is the basis of a transitive uniformity on B^A . We call it the bounded information uniformity.*
2. *The product topology (of the discrete topology on B) is associated to the bounded information uniformity.*

Proof. Point 1 is a straightforward consequence of Proposition A.7. For point 2, we shall use the fact that a basis of the topology associated to the bounded information uniformity is obtained by taking sections $U_\pi|f = \{g \mid (f, g) \in U_\pi\}$ where π varies among deterministic bounded covering and f varies in B^A . If $f \in [u]$ and $u \in \pi$ then $U_\pi|f = [u]$. To conclude, observe that every partial function $u : A \rightarrow B$ with finite domain belongs to some deterministic uniformly bounded covering π . \square

The bounded information uniformity is the right topological tool to characterize functionals using bounded information.

Theorem 4.2. *Endow B with the discrete uniformity and B^A with the bounded information uniformity. A total functional $\Phi : B^A \rightarrow B$ is uniformly continuous if and only if it uses uniformly bounded information.*

Proof. Since we consider the discrete uniformity on the range set B , Φ is uniformly continuous if and only if the inverse image of the diagonal $(\Phi, \Phi)^{-1}(\{(b, b) \mid b \in B\})$ is an entourage of the bounded information uniformity, i.e. contains an entourage U_π where π is a uniformly bounded covering. This means that there is a deterministic uniformly bounded covering π such that, for all $f, g \in B^A$, if $(f, g) \in U_\pi$ then $\Phi(f) = \Phi(g)$. Now, $(f, g) \in U_\pi$

exactly means $f, g \in [u]$ for some $u \in \pi$. Thus, Φ is uniformly continuous if and only if Φ uses deterministic uniformly bounded information. To conclude, apply Corollary 3.8. \square

4.2 Curryfication and linear uniform continuity

Definition 4.3. Endow B^A with the bounded information uniformity. We say that a total functional $\Psi : B^A \rightarrow B^A$ is linearly uniformly continuous if the following condition holds:

$$\exists k \quad \forall \ell \in \mathbb{N} \quad \forall \ell\text{-bounded covering } \pi \quad \exists k\ell\text{-bounded covering } \pi' \\ \forall f, g \in \mathbb{N}^{\mathbb{N}} \quad ((f, g) \in U_{\pi'} \Rightarrow (\Psi(f), \Psi(g)) \in U_{\pi})$$

Proposition 4.4. Let $\Psi : B^A \rightarrow B^A$ be a total functional. Endow B with the discrete uniformity and B^A and $N^{A \cup \{\top\}}$ with the bounded information ones.

1. If Ψ is linearly uniformly continuous then it is uniformly continuous.
2. Ψ is linearly uniformly continuous if and only if $\partial\Psi : B^{A \cup \{\top\}} \rightarrow B$ is uniformly continuous (cf. Definition 1.3).

Proof. Point 1 is obvious. We first prove the \Rightarrow implication in Point 2. Suppose Ψ is linearly uniformly continuous with the constant k . For $a \in A$, let π_a be the deterministic 1-bounded covering of all partial functions $A \rightarrow B$ with domain $\{a\}$. Linear uniform continuity insures that there exists a deterministic k -bounded covering π'_a of B^A such that, for all $u \in \pi'_a$, $f \mapsto \Psi(f)(a)$ is constant on $[u]$. Then $\partial\Psi$ is constant on $[u \oplus (\top, a)]$ where $u \oplus (\top, a)$ is the extension of u to $\text{Domain}(u) \cup \{a\}$ which takes value a on \top . The family τ of all $u \oplus (\top, a)$, where $a \in A$ and $u \in \pi'_a$, is then a deterministic $(k+1)$ -bounded $\partial\Psi$ -family. By Theorem 4.2, $\partial\Psi$ is uniformly continuous.

For the converse \Rightarrow implication, suppose $\partial\Psi$ is uniformly continuous and let τ be a deterministic k -covering of $B^{A \cup \{\top\}}$ such that $\partial\Psi$ is constant on $[v]$ for every $v \in \tau$. Lifting k to $k+1$, we can suppose that every $v \in \tau$ is defined on \top . Let π be a deterministic ℓ -covering of B^A . For each $u = ((a_1, b_1), \dots, (a_p, b_p)) \in \pi$ with $p \leq \ell$ (cf. Notation 1.2), let π'_u be the family of all $u_1 \oplus \dots \oplus u_p$ such that $u_1 \dots, u_p$ are pairwise compatible and $u_i \oplus (\top, a_i) \in \tau$ and the value of $\partial\Psi$ on $[u_i \oplus (\top, a_i)]$ is b_i for all $i = 1, \dots, p$. Set $\pi' = \bigcup_{u \in \pi} \pi'_u$. Since each $u \in \pi$ has a size $\leq \ell$ domain and each $u_i \oplus (\top, a_i) \in \tau$ has a size k domain, we see that each function in π' has a size $\leq k\ell$ domain. Let $f \in \mathbb{N}^{\mathbb{N}}$. There exists a unique $u = ((a_1, b_1), \dots, (a_\ell, b_\ell)) \in \pi$ such that $\Psi(f) \in [u]$. Observe that $b_i = \Psi(f)(a_i)$ for all i . For each $i = 1, \dots, p$ there exists a unique u_i such that $u_i \oplus (\top, a_i) \in \tau$ and $f \oplus (\top, a_i) \in [u_i \oplus (\top, a_i)]$. Since $f(a_i) = b_i$ for all i , we see that $u_1 \oplus \dots \oplus u_p$ is in π'_u hence in π' . Thus,

$f \in [u_1 \oplus \dots \oplus u_p]$. This proves that π' is a deterministic $k\ell$ -covering of B^A . Finally $\Psi(g) \in [u]$ for every $g \in [u_1 \oplus \dots \oplus u_p]$. Thus, $(\Psi, \Psi)(U_{\pi'}) \subseteq U_\pi$ and Ψ is $(\ell \mapsto k\ell)$ -uniformly continuous. \square

5 Effective uniformly bounded information

The effectivization of uniform continuity is the expected one.

Definition 5.1. *A computable total functional $\Phi : B^A \rightarrow B$ is effectively uniformly continuous relative to the bounded uniformity on B^A if the inverse image of the diagonal of $B \times B$ contains a computable uniformly bounded covering of B^A (in other words, if there exists a computable (Φ, k) -family for some $k \in \mathbb{N}$).*

The proof of Theorem 3.7 uses a dichotomy: Φ is constant or not. This dichotomy is used recursively with the $\Phi^{(w)}$'s. Alas, to decide whether $\Phi^{(w)}$ is constant or not requires either the halting problem as oracle or the extra hypothesis that $X_{\Phi, k}$ is computable. Using an oracle leads to non computable functions $\theta_j^{(i)}$'s in the constructor Θ ! And the extra hypothesis about $X_{\Phi, k}$ fails in general for ASM functionals, the ones we want to characterize!

To get computable functions $\theta_j^{(i)}$'s, we give a completely different proof using Ramsey's theorem. The price of this new proof is a super-exponential blow-up from k to a Ramsey number involving k .

Theorem 5.2 (Ramsey's theorem). *Let $c \geq 2$. There exists a primitive recursive function $R : \mathbb{N}^c \rightarrow \mathbb{N}$ such that, for every coloring γ of the unordered pairs of $\{1, \dots, R(n_1, \dots, n_c)\}$ into c colors $1, \dots, c$, there exist a color i and a subset X_γ with n_i elements such that all pairs in X_γ have color i .*

The following Lemma is the key of the proof of Theorem 5.4.

Lemma 5.3. *Let $\Phi : B^A \rightarrow B$ be a total functional using k -bounded information. Let $(v_i)_{i=1, \dots, L}$, $(w_i)_{i=0, \dots, L}$ be sequences in $\mathfrak{F}_k^{A \rightarrow B}$ such that, for all i ,*

- (1) $v_i \in X_{\Phi, k}$ and $\text{Domain}(w_i) = \text{Domain}(v_i)$,
- (2) if $i < j$ then w_i and v_j agree on the common part of their domains,
- (3) w_0, \dots, w_L pairwise agree on the common part of their domains.

Suppose $L \geq R(n_0, \dots, n_k)$, the Ramsey number where $n_c = 1 + \binom{k}{c}(k+1)$ for $c = 0, \dots, k-1$ and $n_k = 2$. Then Φ is constant on $[\vec{w}]$ where $\vec{w} = w_0 \oplus \dots \oplus w_L$.

Proof. For $1 \leq i < j \leq L$ let $\gamma(i, j)$ be the number of elements in $\text{Domain}(v_i) \cap \text{Domain}(v_j)$. This function γ is a coloring of $\{1, \dots, L\}$ into $k + 1$ colors $0, \dots, k$. Since the v_ℓ 's have $\leq k$ elements in their domains, γ is a coloring of pairs in $\{1, \dots, L\}$ into $k + 1$ colors $0, \dots, k$. The hypothesis on L and Ramsey's theorem 5.2 insure the existence of a color c and a subset X of $\{1, \dots, L\}$ having n_c elements (cf. formula (*) supra) on which all pairs have color c . We argue on the color c .

- *Case $c = k$.* Since $n_k = 2$ there are $i < j$ such that $\text{Domain}(v_i) = \text{Domain}(v_j)$. Now, by definition, since $i < j$, v_j and w_i agree on the common part of their domains. Since $\text{Domain}(w_i) = \text{Domain}(v_i)$ we have $w_i = v_i$. Since $v_i \in X_\Phi$, we see that Φ is constant on $[w_i]$ hence is constant on $[\vec{w}]$.

- *Case $0 \leq c < k$.* The v_i 's, $i \in X$, are functions in Φ, k such that the domains of any pair of these functions have an intersection with exactly c points. Let ℓ be the least element of X . For each subset D of $\text{Domain}(v_\ell)$ with c elements, let $X_D = \{i \in X \setminus \{\ell\} \mid \text{Domain}(v_\ell) \cap \text{Domain}(v_i) = D\}$. The X_D 's form a partition of X . Since $n_c = 1 + \binom{k}{c}(k + 1)$ there are $(k + 1)\binom{k}{c}$ points in $X \setminus \{\ell\}$. Since there are $\binom{k}{c}$ possible subsets D , we see that there is a particular subset D such that X_D has at least $k + 1$ elements. In particular, if $i, j \in X_D$, $i < j$, then $\text{Domain}(v_i) \cap \text{Domain}(v_j) = D$. Now, all v_i 's, $i \in X_D$ agree with w_ℓ hence $v_i \upharpoonright D = w_\ell \upharpoonright D$. Proposition 3.13 insures that Φ is constant on $w_\ell \upharpoonright D$. A fortiori, Φ is constant on $[\vec{w}]$. \square

Main Theorem 5.4. *There exists a primitive recursive total function $L : \mathbb{N} \rightarrow \mathbb{N}$ having the following property. Suppose Φ is a computable total functional $B^A \rightarrow B$ and π is a computable (Φ, k) -family for some $k \in \mathbb{N}$. Then there exist elements s_1, \dots, s_k in A , computable total functions $\theta_j^{(i)} : B^{ik} \rightarrow A$, with $i = 1, \dots, L(k)$, $j = 1, \dots, k$, and a computable total function $\omega : B^{(1+L(k))k} \rightarrow B$ such that $\Phi = \Omega_\Theta \upharpoonright B^A$ and π_Θ is a $(\Phi, k(1+L(k)))$ -family, where Θ is the constructor*

$$\Theta = \left(s_1, \dots, s_k, \theta_1^{(1)}, \dots, \theta_k^{(1)}, \dots, \theta_1^{(L(k))}, \dots, \theta_k^{(L(k))}, \omega \right) .$$

2. *The above result is uniform in Φ, π : if Φ and π depend computably on a parameter $m \in \mathbb{N}$ then ω and the s_i 's, $\theta_j^{(i)}$'s are uniformly computable in m .*

Proof. Let $(u_i)_{i \in \mathbb{N}}$ be a computable enumeration of π . Let $L(k)$ be the Ramsey number L of Lemma 5.3. We fix some computable total ordering on A and define by induction on $i \geq 1$ the s_j 's and $\theta_j^{(i)}$'s, together with $\rho_i : B^{ik} \rightarrow \mathbb{N}$. Let $\vec{b} = (b_1, \dots, b_{k(L(k)+1)}) \in B^{k(L(k)+1)}$. Let us denote by $\vec{b}|ik$ the sequence $(b_1, \dots, b_{ik}) \in B^{ik}$. We define inductively the $\theta_j^{(i)}(\vec{b}|ik)$.

- (1) Choose s_1, \dots, s_k (non necessarily distinct) such that $\{s_1, \dots, s_k\} = \text{Domain}(u_0)$.

(2) Let $\rho_1(\vec{b}|k)$ be the least integer ℓ such that u_ℓ and $(s_k, b_k) \oplus \dots \oplus (s_1, b_1)$ agree on the common part of their domains (recall Definition 1.1 and observe the inversion done in the sequence) and let $\theta_1^{(1)}(\vec{b}|k), \dots, \theta_k^{(1)}(\vec{b}|k)$ (non necessarily distinct) be such that $\{\theta_1^{(1)}(\vec{b}|k), \dots, \theta_k^{(1)}(\vec{b}|k)\} = \text{Domain}(u_{\rho_1(\vec{b}|k)})$.

(3) Suppose the $\theta_j^{(i')}(\vec{b}|i'k)$'s, and the $\rho_{\vec{b}|i'k}$'s have been defined for $i' < i$.

Let us write $s_{i'k+j} = \theta_j^{(i')}(\vec{b}|i'k)$. Let $\rho_i(\vec{b}|ki)$ be the least integer ℓ such that u_ℓ and $(s_{ik}, b_{ik}) \oplus \dots \oplus (s_1, b_1)$ agree on the common part of their domains and let $\theta_1^{(i)}(\vec{b}|ik), \dots, \theta_k^{(i)}(\vec{b}|ik)$ (non necessarily distinct) be such that $\{\theta_1^{(i)}(\vec{b}|ik), \dots, \theta_k^{(i)}(\vec{b}|ik)\} = \text{Domain}(u_{\rho_i(\vec{b}|ki)})$.

Let $v_i = u_{\rho_i(\vec{b}|ki)}$ and $w_i = ((s_{ik+k}, b_{ik+k}) \oplus \dots \oplus (s_1, b_1)) \upharpoonright \text{Domain}(v_i)$. Then all conditions of Lemma 5.3 are satisfied: $\text{Domain}(v_i) = \text{Domain}(w_i)$, the w_i 's pairwise agree (due to the inversion in the \oplus sum) and w_i agrees with v_j for $i < j$ by definition of $\rho_j(\vec{b}|jk)$. Thus, Φ is constant on $w_{kL(k)} \oplus \dots \oplus w_0$. Since $\text{Domain}(w_K)$ is the set of the $\theta_j^{(i)}(\vec{b}|ik)$'s, we see that Φ is constant on all $u \in \pi_\Theta$. Finally, we show that the values of Φ are given by some computable ω . Define ω as follows. Let $\zeta : A \rightarrow B$ be any constant function. For $\vec{b} \in B^{k(L(k)+1)}$, let $u = w_{kL(k)} \oplus \dots \oplus w_0$ be as above and let $f_{\vec{b}} = \zeta \oplus u$ and set $\omega(\vec{b}) = \Phi(f_{\vec{b}})$. By the above analysis, we know that Φ is constant on $[u]$. Since $f_{\vec{b}} \in [u]$, this constant value is given by $\omega(\vec{b})$. To conclude, recall that Φ is computable and observe that $f_{\vec{b}}$ is a computable function, uniformly in \vec{b} , hence θ is computable. \square

Corollaries 3.8 and 3.9 have effective versions.

Corollary 5.5. *Suppose $\Phi : B^A \rightarrow B$ is a computable total functional. The following conditions are equivalent.*

- (1) Φ is effectively uniformly continuous.
- (2) Φ admits a computable uniformly bounded Φ -family.
- (3) Φ admits a computable deterministic uniformly bounded Φ -family.
- (4) $\Phi = \Omega_\Theta \upharpoonright B^A$ for some computable constructor Θ as in Definition 3.4.

Corollary 5.6. *If a total functional $\Phi : B^A \rightarrow B$ is computable and effectively uniformly continuous then there exists a computable total functional $\Psi : B^A \rightarrow \mathcal{F}^{A \rightarrow B}$ such that,*

- (a) Ψ is computable and effectively uniformly continuous.
- (b) The range of Ψ is a computable deterministic uniformly bounded Ψ -family and Φ -family, i.e. for all $f \in B^A$, Ψ and Φ are constant on $[\Psi(f)]$.

(c) For all $f \in B^A$, $f \in [\Psi(f)]$.

6 Characterization of ASM functionals

Proposition 6.1. *Let $T = \prod_{i=1, \dots, n} B_i^{A_i}$ be a product of function spaces where the A_i, B_i 's are countable. Let A, B be the disjoint sums of the A_i 's and the B_i 's respectively, so that B^A is a (proper) subset of T . Let $\Phi : T \rightarrow C$ a total functional which uses k -bounded information. Then there exists a total functional $\tilde{\Phi} : B^A \rightarrow C$ which also uses $(k+1)$ -bounded information and is such that $\Phi = \tilde{\Phi} \upharpoonright T$. Moreover, if Φ is computable and admits a computable (Φ, k) -family then so does $\tilde{\Phi}$ is computable and admits a computable $(\tilde{\Phi}, k)$ -family.*

Proof. Let us say that a partial function in $\mathfrak{F}^{A \rightarrow B}$ is bad if it maps some element of A_i outside B_i . Let π be a (Φ, k) -family. We define a $(\tilde{\Phi}, k+1)$ -family from π . Let $u \in \pi$. Interpreted in $\mathfrak{F}^{A \rightarrow B}$, all extensions with bad elements outside $\text{Domain}(u)$ are obtained. The problem is with bad elements inside $\text{Domain}(u)$. We proceed as follows. Identify A with \mathbb{N} . If $c_1, \dots, c_\ell \in \text{Domain}(u)$ and (c_i, d_i) is bad for $i = 1, \dots, \ell$, then, for $x \in A$, let

$$u_{\vec{c}, \vec{d}, x} = u \oplus (c_1, d_1) \oplus \dots \oplus (c_\ell, d_\ell) \oplus (\gamma, x)$$

with $\gamma = \sum_i c_i + \langle c_1, d_1, \dots, c_\ell, d_\ell \rangle$ where $\langle \dots \rangle$ is some coding of sequences of integers by integers. Set $\tilde{\pi}$ be the set of all possible $u_{\vec{c}, \vec{d}, x}$'s. It is easy to see that $\tilde{\pi}$ is a $(\tilde{\Phi}, k+1)$ -family. Finally, this construction is effective. \square

Main Theorem 6.2. *Let $T = \prod_{i=1, \dots, n} B_i^{A_i}$ be a product of function spaces where the A_i, B_i 's are countable. Let $\Psi : T \rightarrow T$ be a total functional. The following conditions are equivalent.*

- (1) Ψ uses uniformly bounded information.
- (2) There exists an ASM on the sorts occurring in T (with a suitable static framework) such that Ψ is exactly the functional associated to this ASM.

Moreover, if Ψ is computable and effectively linearly uniformly computable (i.e. there is some computable uniformly bounded $\partial\Psi$ -family) then the static framework of the ASM consists of computable functions.

Proof. Consider the functional $\tilde{\partial\Psi}$ given by Proposition 6.1 and use Theorems 3.7, 4.2, 5.4 to $\tilde{\partial\Psi}$. \square

References

- [1] Andreas Blass and Yuri Gurevich. Abstract state machines capture parallel algorithms. *ACM Transactions on Computational Logic*, 4(4):578–651, 2003.
- [2] Andreas Blass and Yuri Gurevich. Abstract state machines capture parallel Algorithms: correction and extension. *ACM Transactions on Computational Logic*, 9(3), 32 p., 2008.
- [3] Nicolas Bourbaki. *General topology, Chapters 1–4*. Springer-Verlag, 1998.
- [4] Nachum Dershowitz and Yuri Gurevich. A natural axiomatization of computability and proof of Church’s Thesis. *Bulletin. of Symbolic Logic*, 14(3):299–350, 2008.
- [5] Marie Ferbus and Serge Grigorieff. ASM and operational algorithmic completeness of Lambda Calculus. *LNCS 6300*, 301–327, 2010.
- [6] Serge Grigorieff and Pierre Valarcher. Evolving MultiAlgebras unify all usual sequential computation models. *STACS 2010*, 2009.
- [7] Scott Dexter and Patrick Boyle and Yuri Gurevich. Gurevich Abstract State Machines and Schönage Storage Modification Machines. *JUCS*, 3(4): 279–303, 1997.
- [8] Yuri Gurevich. Reconsidering Turing’s Thesis: towards more realistic semantics of programs. *Technical Report CRL-TR-38-84, EEC Dept, Univ. Michigan*, 1984.
- [9] Yuri Gurevich. A new Thesis. *Abstracts, American Math. Soc.*, 1985.
- [10] Yuri Gurevich. Evolving Algebras: An Introductory Tutorial. *Bul. EATCS*, 43: 264–284, 1991. Reprinted in *Current Trends in Theoretical Computer Science, 1993*, 266–29, World Scientific, 1993.
- [11] Yuri Gurevich. *May 1997 Draft of the ASM Guide*. Tech Report CSE-TR-336-97, EECS Dept, University of Michigan, 1997.
- [12] Yuri Gurevich. The Sequential ASM Thesis. *Bul. EATCS*, 67: 93–124, 1999. Reprinted in *Current Trends in Theoretical Comp. Sc., 2001*, 363–392, World Scientific, 2001.

- [13] Yuri Gurevich. Sequential Abstract State Machines capture Sequential Algorithms. *ACM Transactions on Computational Logic*, 1(1):77–111, July 2000.
- [14] John L. Kelley. *General topology*. Springer-Verlag.

A Uniform structure and topology

To get a topological characterization of functionals using bounded information analog to that in Proposition 2.3, we need to replace continuity by uniform continuity relative to a pertinent uniform structure on the Baire space.

First, let us recall the notion of uniformity on a space and some related classical results, cf. Bourbaki or Kelley’s classical textbooks [3, 14].

Definition A.1. *A uniformity \mathcal{U} on a space S is a family of subsets of $S \times S$ (called entourages) satisfying the following conditions:*

- (1) *Every entourage $U \in \mathcal{U}$ contains the diagonal $\Delta = \{(x, x) \mid x \in S\}$.*
- (2) *\mathcal{U} is a filter: every superset of an entourage is an entourage and the intersection of finitely many entourages is an entourage.*
- (3) *If U is an entourage so is its symmetric $U^{sym} = \{(y, x) \mid (x, y) \in U\}$.*
- (4) *For every entourage U there exists an entourage V such that $V \circ V \subseteq U$ where $V \circ W = \{(x, z) \mid (\exists y) ((x, y) \in V \wedge (y, z) \in W)\}$. In particular, since V contains the diagonal, $V = V \circ \Delta \subseteq V \circ V \subseteq U$.*

The uniformity \mathcal{U} is transitive in case condition (4) can be strengthened to

- (4') *For every entourage U there exists an entourage V such that $V \circ V = U$. One can then suppose V to be symmetric.*

A basis \mathcal{B} of \mathcal{U} is a family of entourages such that every entourage contains an entourage in \mathcal{B}

Example A.2. 1. Suppose $d : S \times S \rightarrow [0, +\infty]$ is a metric on S . For any $r > 0$ let $U_r = \{(x, y) \in S \times S \mid d(x, y) \leq r\}$. Then the family of supersets of the U_r ’s is a uniformity \mathcal{U}_d on S (and the U_r ’s constitute a basis of \mathcal{U}_d).

2. In case d is an ultrametric, i.e. $d(x, z) \leq \max(d(x, y), d(y, z))$ for all $x, y, z \in S$, then $U_r \circ U_r = U_r$ and therefore \mathcal{U}_d is a transitive uniformity.

3. The finest uniformity is the discrete one: it which contains the diagonal (hence all its supersets).

To every uniformity is associated a topology.

Proposition A.3. *Suppose \mathcal{U} is a uniformity on S . For every $x \in S$ let $\mathcal{N}_{\mathcal{U}}(x)$ be the family of sets of the form $\{y \in T \mid (x, y) \in U\}$ for some*

entourage $U \in \mathcal{U}$. Let $\mathcal{T}_{\mathcal{U}}$ be the family of sets $X \subseteq T$ such that $X \in \mathcal{N}_{\mathcal{U}}(x)$ for every $x \in X$.

Then $\mathcal{T}_{\mathcal{U}}$ is the family of open sets of a topology on X and $\mathcal{N}_{\mathcal{U}}(x)$ is the family of neighborhoods of x with respect to the topology $\mathcal{T}_{\mathcal{U}}$.

In the same way that topology is the right framework to deal with continuity, the notion of uniformity is the right one to deal with uniform continuity.

Definition A.4. Suppose \mathcal{U} and \mathcal{V} are uniformities on the respective spaces S and T . A map $F : S \rightarrow T$ is uniformly continuous with respect to \mathcal{U} and \mathcal{V} if $(F \times F)^{-1}(V) \in \mathcal{U}$ for every $V \in \mathcal{V}$, i.e. the inverse image by $F \times F$ of an entourage of \mathcal{V} is an entourage of \mathcal{U} . In other words, for every entourage $V \in \mathcal{V}$ there exists an entourage $U \in \mathcal{U}$ such that if x, y are U -close, i.e. $(x, y) \in U$, then $F(x), F(y)$ are V -close, i.e. $(F(x), F(y)) \in V$.

This definition extends the usual one with metric spaces.

Proposition A.5. If (S, d_S) and (T, d_T) are metric spaces then a map $F : S \rightarrow T$ is uniformly continuous with respect to the uniformities \mathcal{U}_{d_S} and \mathcal{U}_{d_T} (cf. Example A.2 supra) if and only if F is uniformly continuous in the sense of metric spaces: for every $\varepsilon > 0$ there exists $\eta > 0$ such that, for all $x, y \in S$, if $d_S(x, y) \leq \eta$ then $d_T(F(x), F(y)) \leq \varepsilon$.

As expected, uniform continuity implies continuity.

Proposition A.6. If $F : S \rightarrow T$ is uniformly continuous with respect to \mathcal{U} and \mathcal{V} then F is continuous with respect to the associated topologies $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{V}}$.

Transitive uniformities can be characterized via partitions.

Proposition A.7. To any partition $\sigma = (S_{\alpha})_{\alpha \in A}$ of a space S we associate the set $V_{\sigma} = \bigcup_{\alpha \in A} S_{\alpha} \times S_{\alpha}$. Let \mathcal{U} be a uniformity on S and Σ be the family of all partitions σ such that $U_{\sigma} \in \mathcal{U}$. The uniformity \mathcal{U} is transitive if and only if $\{V_{\sigma} \mid \sigma \in \Sigma\}$ is a basis of \mathcal{U} .

Proof. Suppose $V \subseteq S \times S$ contains the diagonal and is symmetric. Observe that V satisfies $V \circ V = V$ if and only if V is an equivalence relation on S if and only if $V = U_{\sigma}$ for some partition σ (namely the partition constituted by equivalence classes of V). \square

Finally, let us mention the following property of transitive uniformities.

Proposition A.8. Suppose \mathcal{U} is a transitive uniformity on S . A basis of the topology associated to \mathcal{U} is the family of all pieces S_{α} of all partitions σ of S such that U_{σ} is in \mathcal{U} .

Proof. Observe that $S_\alpha = \{y \mid (x, y) \in S_\alpha \times S_\alpha\} = \{y \mid (x, y) \in U_\sigma\}$. In particular, if $x \in S_\alpha$ then S_α is a neighborhood of x . Thus, S_α is a neighborhood of each of its points hence is open. Conversely, if $X \subseteq S$ is open then, for every $x \in X$, there exists a partition σ such that U_σ is in \mathcal{U} and a piece S_α of σ such that $x \in \{y \mid (x, y) \in U_\sigma\} \subseteq X$. Thus, $x \in S_\alpha \subseteq X$. Since x is any element of X , this shows that X is a union of such sets S_α 's. Thus, any open set is a union of S_α 's. This means that the S_α 's constitute a topological basis. \square